

**PRACTICE #1: Make sense of problems and persevere in solving them.**

- What constitutes a cognitively demanding task?

*Lower-level demands (memorization)*

- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory.
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.

*Lower-level demands (procedures without connections to meaning)*

- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers instead of on developing mathematical understanding.
- Require no explanations or explanations that focus solely on describing the procedure that was used.

*Higher-level demands (procedures with connections to meaning)*

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

*Higher-level demands (doing mathematics)*

- Require complex and non-algorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

Arbaugh, F., & Brown, C. A. (2005). Analyzing mathematical tasks: a catalyst for change? *Journal of Mathematics Teacher Education*, 8, p. 530.

- What can I do in my classroom to attend to multiple entry points, different solutions paths, and appropriate time? How do these features support student learning?
  - If a task meets the features of high cognitive demand (doing mathematics or procedures with connection to meaning), it should inherently have multiple entry points and solution paths.
  - The following journal article summarizes a study where tasks were well developed but problems with implementation reduced the cognitive demand of the task (one of the biggest factors was poor time allotment):
    - Stein, M. K., & Henningsen, M. (1997). Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education* , 28 (5), 524-549.
  - Stein also has a text for teachers and teacher educators interested in synthesizing their current practice with new mathematics standards. Presented are cases of mathematics instruction drawn from research of nearly 500 classroom lessons. Readers will gain insight about how to foster a challenging, cognitively rich, and exciting classroom climate that propels students toward a richer understanding of mathematics.
    - Stein, M. K. (2000). *Implementing Standards-Based Instruction: A Casebook for Professional Development*. New York: Teachers College Press.
- Plan task-specific questions to explicitly integrate meta-cognition. The following list is adapted from the link below, retrieved February 2013, from <http://www.scribd.com/doc/23035034/Metacognitive-questions>
  - Reflective and Reasoning Questions:
    - How did you decide what to include?
    - Why did you write that/put that there?
    - How did you start off?
    - What did you find the most difficult? How did you tackle it?
    - Did you use any images in your head to help you?
    - How did you work together? Did it help?
    - How did you decide to leave information out?
    - What assumptions have you made?
    - What connections have you made? What makes a good connection?
    - Did you have a plan and did you have to change it?
    - Has anyone got an answer you like? Why?
  - Extension Questions:
    - Are features of this problem more important than others?
    - Where could you use what you have learned today with previous problems we have looked at?
    - What would be a different situation where your solution path would also work?

- An additional list of effective questions for mathematical thinking, developed by PBS TeacherLine, can be found by following the link below, retrieved February 2013, from [http://mason.gmu.edu/~jsuh4/teaching/resources/questionsheet\\_color.pdf](http://mason.gmu.edu/~jsuh4/teaching/resources/questionsheet_color.pdf).
- The chart below provides different types of meta-cognitive questions; some of these could be appropriate for students to ask one another as well. Retrieved February 2013, from <http://mason.gmu.edu/~jsuh4/teaching/resources/mathjournals.pdf>

<b>Reflecting on Problem Solving Clear Communication</b>	<b>Respectful Communication</b>	<b>Flexible Thinking</b>	<b>Persistence</b>
What math words could help us share our thinking about this problem? Choose 2 and explain what they mean in your own words.	Did someone else solve the problem in a way you had not thought of? Explain what you learned by listening to a classmate.	What other problems or math topics does this remind you of? Explain your connection.	What did you do if you got “stuck” or felt frustrated?
What could you use <i>besides words</i> to show how to solve the problem? Explain how this representation would help someone understand.	Did you ask for help or offer to help a classmate? Explain how working together helped solve the problem.	Briefly describe at least 2 ways to solve the problem. Which is easier for you?	What helped you try your best? <i>or</i> What do you need to change so that you can try your best next time?
If you needed to make your work easier for someone else to understand, what would you change?	What helped you share and listen respectfully when we discussed the problem? <i>or</i> What do you need to change so that you can share and listen respectfully next time?	What strategies did you use that you think will be helpful again for future problems?	Do you feel more or less confident about math after trying this problem? Explain why.

**PRACTICE #2: Reason abstractly and quantitatively.**

- What is a *realistic* context?
  - We purposefully do not use the term *real-world* here because it is difficult to have a truly real-world context that can be reduced to something appropriate for the mathematics; thus, in saying *realistic* we mean that the situation can be imagined – a student could place themselves in the context
  
- NCTM describes key elements of reasoning and sense making specific to functions and representations (NCTM. (2009). Reasoning with Functions. In *Focus in High School Mathematics-Reasoning and Sense Making* (pp. 41-53). Reston: National Council of Teachers of Mathematics).
  - Representing functions in various ways – including tabular, graphic, symbolic (explicit and recursive), visual, and verbal
  - Making decisions about which representations are most helpful in problem-solving circumstances
  - Moving flexibly among those representations
  
- NCTM breaks down the knowledge of functions into five essential understandings. The fifth essential understanding pertains to multiple representations. The descriptors given below are major areas of focus under Big Idea 5. These essential understandings further describe what it means for a student to have flexibility with representations when studying functions (found in NCTM. (2010). *Developing Essential Understanding of Functions Grades 9-12*. Reston: National Council of Teachers of Mathematics).
  - Essential Understanding 5a. Functions can be represented in various ways, including through algebraic means (e.g., equations), graphs, word descriptions and tables.
  - Essential Understanding 5b. Changing the way that a function is represented (e.g., algebraically, with a graph, in words, or with a table) does not change the function, although different representations highlight different characteristics, and some may show only part of the function.
  - Essential Understanding 5c. Some representations of a function may be more useful than others, depending on the context.
  - Essential Understanding 5d. Links between algebraic and graphical representations of functions are especially important in studying relationships and change.

- Robin Rider gave instructional and assessment recommendations for teachers to improve student fluency with representations (Rider, R. (2007). Shifting from Traditional to Nontraditional Teaching Practices using multiple representations. *Mathematics Teacher*, 100 (7), 494-500).
  - Vary the representation with introduction of new concepts – don't always present using the symbolic form... change between tabular, graphical, or symbolic
  - Create class discussion around strengths and weaknesses of different representations
  - Integrate different representations on assessments
  - Discussion around representation should include exploration of invariance
  - Use technology for further exploration of representations in a less tedious way
  
- Knuth looked at the specific struggles of students to connect and use different representations. Knuth's observations, given below, could help a teacher look for student misconceptions (Knuth, E. J. (2000). Understanding Connections between Equations and Graphs. *The Mathematics Teacher*, 93 (1), 48-53).
  - Even when a graphical approach was significantly more efficient, the majority of students chose algebraic methods
  - When prompted to describe an additional solution process, many students were unable to recognize graphs as a viable path to a solution (17% of students gave an alternative solution method)
  - Students were unable to verify solutions using graphs; in fact, most students did not see the graphs as relevant at all in answering the questions
  - Students seem to have developed a ritualistic approach to finding solutions algebraically
  - Students can really only move in one direction with representations: from equation to graph

- The chart shown below could help in determining the appropriateness of different representations, as well as what features to attend to when posing questions to students (Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In NCTM, *The Roles of Representation in School Mathematics* (pp. 173-185). Reston: The National Council of Teachers of Mathematics, Inc).

Representation	Uses	Advantages	Disadvantages
Verbal	<ul style="list-style-type: none"> <li>▪ problem posing</li> <li>▪ interpretation of solution</li> </ul>	<ul style="list-style-type: none"> <li>▪ tool for solving problems</li> <li>▪ connects mathematics and other domains</li> </ul>	<ul style="list-style-type: none"> <li>▪ language use can be ambiguous or misleading</li> <li>▪ less universal</li> </ul>
Numerical	<ul style="list-style-type: none"> <li>▪ determine numbers to understand problem</li> </ul>	<ul style="list-style-type: none"> <li>▪ gives entry to a problem</li> <li>▪ helps to investigate cases</li> </ul>	<ul style="list-style-type: none"> <li>▪ no generality</li> <li>▪ some important aspects of problem or solution may not be visible</li> </ul>
Graphical	<ul style="list-style-type: none"> <li>▪ provide picture for a function of a variable</li> </ul>	<ul style="list-style-type: none"> <li>▪ visual</li> <li>▪ intuitive for students</li> </ul>	<ul style="list-style-type: none"> <li>▪ lacks accuracy</li> <li>▪ does not include all parts of domain and range</li> </ul>
Algebraic	<ul style="list-style-type: none"> <li>▪ general representation of a pattern</li> </ul>	<ul style="list-style-type: none"> <li>▪ able to manipulate</li> </ul>	<ul style="list-style-type: none"> <li>▪ can obstruct meaning or nature of the problem</li> <li>▪ does not lend to interpretation of results</li> </ul>

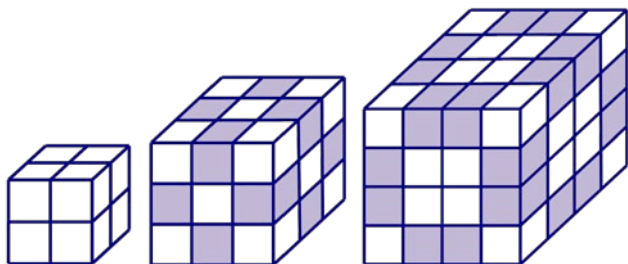
- The following three pages give examples of tasks involving developing and linking representations, as well as articulating connections.

**PAINTED CUBES**

(Adapted from Driscoll, M. (1999). *Fostering Algebraic Thinking*. Portsmouth: Heinemann.)

Each larger cube below is made up of smaller cubes. Someone decided to create a pattern by painting the smaller cubes that have ONLY two exposed faces in a different color. Examine the pattern and work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

1. Discuss how many smaller cubes will be painted in the different color for the next larger cube.
2. Create a table relating the cube # to the total # of cubes with two faces painted in a different color.
3. Determine an equation relating the cube # to the total # of cubes with two faces painted in a different color; write your equation using function notation and explain how you arrived at your equation.
4. Create a graph showing the relationship between the cube # to the total # of cubes with two faces painted in a different color.
5. Select one coordinate pair from your table. Identify the same coordinate pair in the figures, the equation, and the graph.
6. Explain the relationship between the table, equation, and graph.



**COMPARING SAVINGS PLANS**

(Adapted from Coulombe, W. N., & Berenson, S. B. (2001). Representations of patterns and functions - tools for learning. In NCTM, *The Roles of Representation in School Mathematics* pp. 166-172). Reston: The National Council of Teachers of Mathematics, Inc)

Many high school students work during the summer and put some of their money into savings. This is very helpful when you go away to college and need some spending money. The easiest way to save money is to put a little bit away each week. Four different students - Brittni, Steven, Kyler, and Erik - have different savings plans. Review each plan below and then answer the questions that follow.

**BRITJNI**

Brittni had some money in savings from last summer. This past summer she put away a little bit of money each week. The table below shows how much Brittni had in her savings account after a given number of weeks.

week	1	2	3	4	5	6
Deposit Amount	250	290	330	370	410	450

**ERIK**

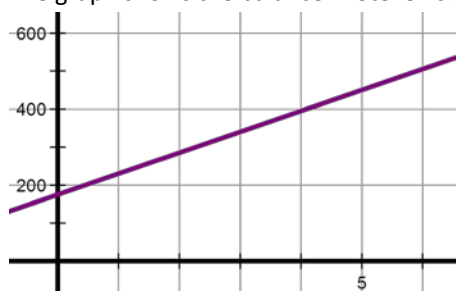
Erik's job automatically deposits a designated amount into his account every week but he can't remember how much. If the balance on his account is  $B(x)$  where  $x$  is in weeks, Erik does know on two different occasions what the balance was:  $B(4) = 550$  and  $B(7) = 730$ .

**KYLER**

Kyler's balance,  $B(x)$ , in his savings account can be represented  $B(x) = 625 - 18x$  where  $x$  stands for the number of weeks.

**STEVEN**

The graph shows the balance in Steven's account based on the number of weeks.



**GROUP QUESTIONS**

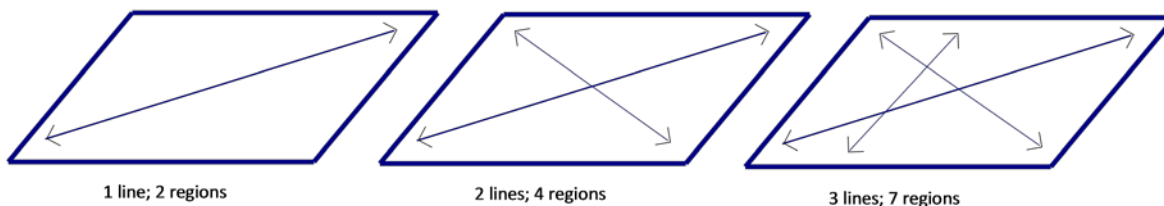
1. Which person do you think has the best savings plan? Justify your answer using an algebraic representation and a second representation of your choice.
2. How does each savings plan compare?
3. Who will have the most money in their savings account at the end of an 8 week summer? Explain how you know.



**PATTERNS, PLANE AND SYMBOL**

(NCTM. (2009). Reasoning with Functions. In *Focus in High School Mathematics-Reasoning and Sense Making* (pp. 41-53). Reston: National Council of Teachers of Mathematics)

**Task:** Develop a symbolic representation for a function that produces the number of regions in a plane formed by intersection lines such that no two lines are parallel and no more than two lines intersect in the same point, as shown in the figure.



**Method 1:** After exploring a number of cases students might produce a table of values for the number of lines and the number of regions. They can then use the table to develop a recursive definition of a function.

**Method 2:** Students could use a geometric approach with coins or tiles to create a pattern. Use the configuration of the pattern students might be able to determine the explicit form of the function.

**Method 3:** By applying technology to numeric and graphical reasoning, students may enter a number of ordered pairs from the table into a graphing calculator and examine a scatterplot of the pairs to the conjecture that the relationship is quadratic. Students could determine a regression equation and then test ordered pairs from the table.

**Method 4:** The teacher could ask students to focus on the differences between consecutive terms of the sequence of total regions. By applying algebraic reasoning, students may examine the data and observe that the function is quadratic because the first differences are linear so the second differences are constant. Then students could write a system of equations using the quadratic form and three ordered pairs.

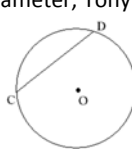
**PRACTICE #3: Construct viable arguments and critique the reasoning of others.**

- The following links give information on constructing arguments
  - [www.learner.org](http://www.learner.org): This website walks through some concrete examples of how to introduce conjectures and simple proofs in class. The material is organized by grade levels and includes general numerical conjectures as well as geometric conjectures. The site also provides reflection questions for the teacher in thinking about the strategies of instruction and evaluation.
    - Grades 3-5. Retrieved February 2013, from [http://www.learner.org/courses/teachingmath/grades3\\_5/session\\_04/index.html](http://www.learner.org/courses/teachingmath/grades3_5/session_04/index.html)
    - Grades 6-8. Retrieved February 2013, from [http://www.learner.org/courses/teachingmath/grades6\\_8/session\\_04/index.html](http://www.learner.org/courses/teachingmath/grades6_8/session_04/index.html)
    - Grades 9-12. Retrieved February 2013, from [http://www.learner.org/courses/teachingmath/grades9\\_12/session\\_04/index.html](http://www.learner.org/courses/teachingmath/grades9_12/session_04/index.html)
  - Henri Picciotto has generously put up a lot of resources on the web regarding his school's Geometry course, specifically on the development of conjectures and proof reasoning. The site links to the three complete Geometry units that appear to be well-written and very thoughtful.
    - Geometry proofs classroom material: <http://www.mathedpage.org/proof/math-2.html>
  - Von Hiele levels are a series of stages in the development of formal geometric logical arguments. This group of links collectively show how teachers can implement geometry lessons while keeping in mind students' cognitive learning needs.
    - How to teach geometry proofs retrieved February 2013, from <http://www.homeschoolmath.net/teaching/geometry.php>
    - Illustration of Von Hiele levels retrieved February 2013, from [http://math.youngzones.org/van\\_hiele.html](http://math.youngzones.org/van_hiele.html)
    - Specific examples retrieved February 2013, from [http://investigations.terc.edu/library/bookpapers/geometryand\\_proof.cfm](http://investigations.terc.edu/library/bookpapers/geometryand_proof.cfm)

- Here is an interesting strategy for increasing student-to-student dialogue within the classroom, via the implementation of “dialogue journals” that get exchanged between students for written feedback.
  - A dialogue journal is an informal written conversation between two or more people (student-student or student-teacher) about topics of mutual interest. Dialogue journals provide students with a meaningful writing activity that is engaging because it involves other students. These written conversations reinforce learning while forming bonds between students that can provide a foundation for later cooperative learning activities. Read more on TeacherVision retrieved February 2013, from <http://www.teachervision.fen.com/writing/skill-builder/48885.html#ixzz2KcrPZNdJ>
  - Implementation of dialogue journals: <http://www.teachervision.fen.com/writing/skill-builder/48885.html>
  - Potential benefits of dialogue journals: <http://www.ucalgary.ca/ieill/Joyce+Bainbridge>
  
- Teaching students to recognize solid vs. weak arguments is a challenging task. The resources below address how to evaluate student arguments.
  - Habits of Mind. Retrieved February 2013, from <http://www2.edc.org/cme/showcase/HabitsOfMind.pdf> (Pg. 11 - 12)
  - *CME Project* textbooks from Educational Development Center (EDC) model a series of “episodes” of conversations between students throughout the books that expose readers to new concepts through proof-like reasoning constructed in student-friendly language. An example, which could be used for classroom discussion, is shown here.

*Minds in Action – Episode 17*

Looking at the definition of diameter, Tony wonders about chords.



**Tony** I once heard that the diameter is the longest chord you can draw for a given circle. Did you know that, Sasha?

**Sasha** As a matter of fact, I did! I think I can prove it. Let’s see.

**Tony** Well, make life simple and start with the first circle at the bottom of the last page. You already have a chord,  $CD$ .

**Sasha** Right! All we have to do is look for triangles, and I love triangles! Connect  $C$ ,  $D$ , and  $O$ . Now I remember that in a triangle the sum of two sides is always greater than the third one. So  $CD < CO + OD$ .

**Tony** You’re brilliant! I know what to do now. I just noticed that  $CO$  and  $OD$  are two radii, so their sum is equal to the diameter. So we’ve proven that any chord is shorter than a diameter.

Benson, Jean, et. al. (2009). *Geometry. CME Project*, 392-393.

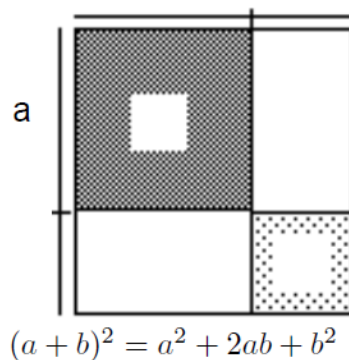
- Here is an example of a challenging problem that pushes student thinking and argument-formation. In the process of arriving at the solution, students should be using a mixture of deduction and experimentation. In the end, the proof is based largely on students' construction, and students should be confident of their results (taken from <http://jwilson.coe.uga.edu/Situations/Framework.Jan08/articles/Cuoco1996HabitsOfMind.pdf>).
- *A square birthday cake is frosted on top and on the four sides. How should it be cut for 7 people if everyone is to get the same amount of cake and the same amount of frosting?*

**PRACTICE #4: Model with mathematics**

- What does it mean to model with mathematics?
  - In the document “Habits of Mind” by Al Cuoco, et. al. mathematical modeling is defined as the attempt to abstract specific mathematical cases into a more general understanding of the concept or behavior.
    - “Getting good at building and applying abstract theories and models comes from immersion in a motley of experiences; noticing that the sum of two squares problem connects to the Gaussian integers comes from playing with arithmetic in both the ordinary integers and in the complex numbers *and* from the habit of looking for similarities in seemingly different situations. But, experience, all by itself, doesn’t do it for most students. They need explicit help in what connections to look for, in how to get started.”
    - This same document outlines techniques for building abstractions and models, which are techniques that should be taught explicitly as mathematical habits to students who are learning to model.
      - ◇ Model objects and changes with functions
      - ◇ Look for multiple perspectives (graphical, algebraic, arithmetic) and to find ways to combine those perspectives to reach deeper conclusions and connections.
      - ◇ Mix deduction and experimentation.
  - “Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.”

- The Common Core Standards give 6 stages within a complete cycle of modeling:
  - 1) Identifying variables in the situation and selecting those that represent essential features,
  - 2) Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables,
  - 3) Analyzing and performing operations on these relationships to draw conclusions,
  - 4) Interpreting the results of the mathematics in terms of the original situation,
  - 5) Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
  - 6) Reporting on the conclusions and the reasoning behind them.
  
- These tips, retrieved February 2013, from <http://serc.carleton.edu/introgeo/models/HowToUse.html> are for introducing mathematical modeling in a science class, but they are useful tips for general mathematical modeling of physical situations.
  - Keep the activity as interactive as possible. When you find that you're spending a majority of your time lecturing to the students about what to do or how things work, try to think of ways you can get them working through ideas in groups, lab, interactive lectures, etc.
  
  - Including students in the development process and/or providing opportunities for them to experiment with the model or modify it can increase students' understanding of the model and its relationship to the physical world.
  
  - Creating opportunities for students to analyze and comment on the models behavior increases their understanding of the relationships between different inputs and rates.
  
  - Creating opportunities for students to validate the model, i.e. compare model predictions to observations, increases their understanding of its limits.
  
  - Stress that models are not reality and that a model's purpose is to help bridge the gap between observations and the real world. An important reason to use a model is that you can perform experiments with models without harming the system of interest.

- Make sure that students think about the underlying assumptions of a model and the domain of applicability. Try to ask questions that can help check their understanding. For example, simple exponential growth assumes that the percent growth rate remains fixed and in real world systems it only applies for so long before the system becomes overstressed. Having students identify underlying assumptions of a model and their domain of applicability can help them gain an appreciation of what a model can and cannot do.
- Models can be used to explore "What-if" scenarios. "What if Atmospheric CO<sub>2</sub> doubles?" is a common example for a climate model.
- A key part of modeling is visualization. Here is an example for geometric modeling of an algebraic concept, from "Habits of Mind" by Al Cuoco, et. al. p.7.



- Other examples of modeling, as provided by the Common Core outline, include:
  - Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
  - Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
  - Designing the layout of the stalls in a school fair so as to raise as much money as possible.
  - Analyzing stopping distance for a car.
  - Modeling savings account balance, bacterial colony growth, or investment growth.
  - Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
  - Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
  - Relating population statistics to individual predictions.

- EDC has provided some additional information about modeling, retrieved February 2013, from [http://thinkmath.edc.org/index.php/Model\\_with\\_mathematics](http://thinkmath.edc.org/index.php/Model_with_mathematics).
  - “Figuring out where to be to catch a ball is not a paper-and-pencil calculation, but it certainly does involve attention to and rough quantification of the speed, direction, and position of both the ball and oneself, as well as other mathematical ideas.”
  - “Doing a KenKen® puzzle is also an application of raw arithmetic skills to an everyday life problem, in the honest sense that puzzling is very much an important part of the life of a child (and important enough even in the lives of adults to assure that even supermarkets stock puzzle books).”
  - “One intent of this standard is to ensure that children see, even at the earliest ages, that mathematics is not just a collection of skills whose only use is to demonstrate that one has them. Even puzzles suffice for that goal.”
  - “What’s important here is not the context that’s used, but the kind of thinking it requires.”
  
- Some problems that connect math to various physical phenomena, with real-world situations that involve mathematical modeling, are found at the website referenced here. One example is given. Retrieved February 2013, from <http://www.math.montana.edu/frankw/ccp/modeling/topic.htm>
  - The thermometer stands at 10 degrees Fahrenheit and the weatherpeople are talking about a wind chill of twenty degrees below zero. Your students come into class shivering and stamping snow from their feet. This is a great opportunity to talk about mathematical modeling -- what does the wind chill mean? -- will the temperature of the water or a person drop below the temperature indicated by a thermometer when the wind is blowing? -- or will it just cool to the same temperature faster? At the front of your room you have equipment set up to explore the effects of temperature and wind.
  
- Additional problem examples, that teachers have used in their classrooms, can be found at the website referenced here. Some examples are given below. Retrieved February 2013, from <http://www.indiana.edu/~iucme/mathmodeling/lessons.htm>.
  - Setting: Pepsi has a game where there is a letter found under each bottle cap. The consumers collect the letters trying to spell out P-E-P-S-I. The probability of getting each letter is not the same. There is a 50% chance that you get a “P”; the chance of getting an “S” is 30.5%; the chance of getting an “I” is 19% and a 0.5% chance of getting an “E”. Task: Using your Random Number Generator on your calculator, simulate the purchasing of Pepsi Bottles and determining what letter you received. Theoretically, how many bottles would you need to purchase in order to spell out PEPSI?



- Background: HIV testing is beginning to be routine for all pregnant women. There is even some talk of making it a mandatory test. If a pregnant woman tests positive for HIV, she is pressured into taking the drug AZT, which is supposed to reduce the risk of transmission to the child at birth. She may be encouraged to have a cesarean, she will be told not to breastfeed and she may be legally required to give her child pharmaceutical drugs as well. However, the drugs are dangerous as they specifically prevent the formation of DNA chains, which could damage a growing fetus or child. If a woman refuses treatment when she has a positive result on her test, she may be taken to court to be forced to use the medications. Question: If pregnant women or people in general, are required to take these tests, how likely is it that they actually have the disease if they test positive? Based on your findings, what would your advice be to the CDC (Center for Disease Control) be in regards to making people take the medication?

Data:

Of 100,000 people tested, these were the results:

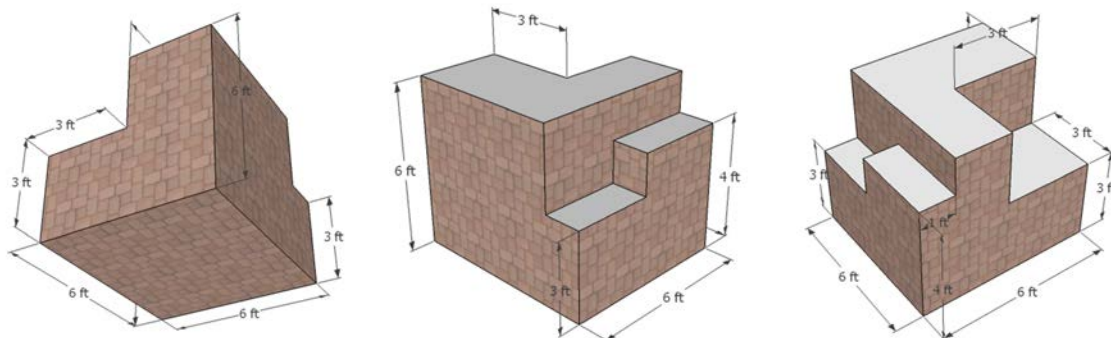
	HIV infected	Not HIV infected
Positive Results	495	995
Negative Results	5	98,505

- **Setup:** You have been hired to work for the Acme Daredevil Entertainment Company. This company provides rock climbing, sky diving, “extreme skiing” and cliff diving adventures to the public. However, to keep up with the market, the board of directors decided to add bungee jumping. **Problem:** Initially each group works with just one action figure. The task is to determine the ultimate length, or the number of rubber bands that can be used with our action figure at any given height and not cause any type of injury or fatality—but that allows you action figure to come as close to the floor as possible. We want to thrill her, not kill her!

**PRACTICE #5: Use appropriate tools strategically**

- There are many free online manipulatives and games that can be used to support a task or to gain a deeper understanding of functions.
  - NCTM Illuminations: This website has many resources and lesson ideas that allow for the integration of various teaching tools into functions lessons.  
<http://illuminations.nctm.org/>
  - National Library of Virtual Manipulatives: Features online manipulatives that can be used as learning tools. <http://nlvm.usu.edu/en/nav/vlibrary.html>
  - Smart Skies: This game was developed by NASA to help students with their understanding of linear functions. <http://www.smartskies.nasa.gov/>
  - Equations of Attack: In this game, students try to sink their opponent's ship using their knowledge of linear functions.  
<http://illuminations.nctm.org/LessonDetail.aspx?id=L782>
  - Exploring Linear Functions: This lesson includes an online manipulative that allows students to systematically change the slope and y-intercept of a linear function and to observe patterns that follow from their changes.  
<http://www.nctm.org/standards/content.aspx?id=26790>
  - Function Matching: Students can use this manipulative to see if they can produce equations that match up with a graph of a given function.  
<http://illuminations.nctm.org/ActivityDetail.aspx?ID=215>
  
- The information below about estimation, retrieved February 2013, is from [http://arb.nzcer.org.nz/supportmaterials/maths/concept\\_map\\_estimation.php](http://arb.nzcer.org.nz/supportmaterials/maths/concept_map_estimation.php).
  - When do we estimate? Students need to be taught how and when to estimate effectively. There are three cases in which estimation is useful:
    - There is no need to have an exact answer. An estimate is good enough: for example *"Do I have enough money?"*
    - There is not enough information to get an exact answer: for example, *"About how many times will my heart beat in an hour?"*
    - To check if the answer from a calculation is sensible.
  
  - What are some strategies of estimation?
    - **Reformulation**, which changes the numbers that are used to ones that are easy and quick to work with.
    - **Compensation**, which makes adjustments that lead to closer estimates. These may be done during or after the initial estimation.

- **Translation**, which changes the mathematical structure of the problem (e.g. from addition to multiplication). Changing the form of numbers so that it alters the mathematical structure of the problem is also translation (Example: 26.7% of \$60 requires multiplication, but this is about 1/4 of \$60, which uses division).
- The lesson plan provided through this link provides examples on how to teach estimation in early grades:  
[http://www.nsa.gov/academia/files/collected\\_learning/elementary/arithmetric/reasonable\\_estimates.pdf](http://www.nsa.gov/academia/files/collected_learning/elementary/arithmetric/reasonable_estimates.pdf).
- At the secondary level, besides using estimation to check answers, students regularly have trouble graphing an estimated location of a “nasty” point in the coordinate plane. Teachers at earlier grades can address this issue by asking students to plot points along estimated locations on a number line.
- There are numerous calculator and computer programs that allow students to explore concepts and deepen their understanding at the secondary level. Some examples and resources are provided below.
  - TI offers a litany of free resources for teachers found at:  
<http://education.ti.com/calculators/downloads/US/Activities/>; you can search by level or topic – activities are available for both the TI-84 Plus family and the nSpire
  - Google sketch-up is a free download that allows you to create 3-D images. You can view from any perspective for investigations with surface area and volume. The download link is: <http://google-sketchup.soft32.com/free-download?lp=adwords&gclid=CLGwi7aW17UCFcdxQgodIDQA3Q>
    - Below is an example of one image made in sketch-up shown from different perspectives.

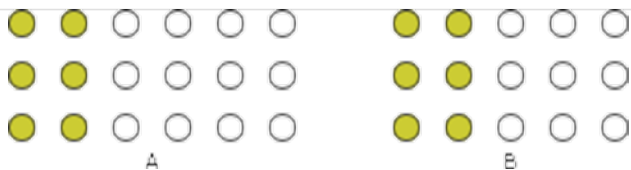


- Geogebra is another free resource that students and teachers can download for their own use. An example of geogebra's capability is to create a function, define parameters within the function, and explore how changing the value of the parameters impacts the various features of a function. Geogebra also has animation features that allow students to observe changes in features with an animation. The link to download is: <http://www.geogebra.org/cms/>. They also provide numerous classroom lessons and resources ready for use!
- Geometer Sketchpad can be used in much the same way as Geogebra. You can define and alter parameters as well as animate. Just as with Geogebra, numerous lessons are provided from Sketchpad. One advantage with Sketchpad is the ability to using inking on a tablet to write on the image. Also, sketchpad has a great feature where you can import an image beneath a graph. Sketchpad can be found at <http://www.dynamicgeometry.com/>. It is not a free resource.
- Fathom is an excellent resource for statistics. You can collect data, show distributions, and make predictions very quickly and easily. Again, Fathom is not a free resource.

**PRACTICE #6: Attend to precision**

- Resources that focus on precision within functions:
  - Green Globes: This game requires students to destroy globs by creating functions that pass through them. <http://www.greenglobs.net/>
  - Exploring Linear Functions: This lesson includes an online interactive tool that allows students to systematically change the slope and y-intercept of a linear function and observe patterns that follow from their changes. Retrieved February 2013 from <http://www.nctm.org/standards/content.aspx?id=26790>
  - Precise Mathematical Language: Jill Kranda conducted an action research study in her middle school classroom. One of her findings is given below. Retrieved February 2013 from <http://scimath.unl.edu/MIM/files/research/KrandaJ.pdf>.
    - “Students resist using precise mathematical language in their solutions because it is not natural to them; therefore, specific vocabulary instruction and repetition in using the vocabulary during instruction is necessary to make using the language more natural for the students.”
  - Pi Filling, Archimedes Style: Students can explore how precision affects their results as they use this method of finding the digits of pi. <http://illuminations.nctm.org/LessonDetail.aspx?id=L714>
- Some cautions for attending to precision follow. These could be helpful guidelines for students (retrieved July 2011 from [http://www.college-algebra.com/essays/writing\\_mathematics\\_correctly.htm](http://www.college-algebra.com/essays/writing_mathematics_correctly.htm)):
  - Use complete sentences, correct grammar, and correct spelling.
  - Symbols (like the + symbol) that have a specific mathematical meaning are reserved for mathematical use.
  - Many mathematical adjectives and nouns have precise mathematical meanings, and an English synonym will not serve as a replacement. For example, "element" and "part" are not interchangeable when referring to an element of a set.
  - Look at examples of writing in the textbook, and try to emulate the style.
  - Two mathematical expressions or formulas in a sentence should be separated by more than just a space or by punctuation; use at least one word.
  - Words have meanings: be aware of them. For example, an **equation** has an equal sign in it. An **expression** is an algebraic combination of terms with no equal sign.
  - Don't use abbreviations.
  - Don't end a line with an equal sign or an inequality sign.

- Honor the equal sign.
  - Use different letters for different things.
  - Define all terms or variables.
  - Once a variable has been assigned a meaning, do not re-use it with a different meaning in the same context.
  - Avoid the use of imprecise terms.
  - Be sure that the use of a term agrees with the definition of that term.
  - Conclude the solution of a problem with a clear and complete statement of the conclusion.
- Professor Hung-Hsi Wu from Berkeley’s mathematics department has written copiously on the issue of teacher precision in the classroom. He considers precision one of the basic characteristics of the essence of mathematics that is important for K-12 teaching, along with Definitions, Reasoning, Coherence, and Purposefulness. He defines precision as “Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.” Wu asserts that “Precision in the vocabulary is necessary because it is only through this vocabulary that we can transcribe intuitive spatial information into precise mathematics, and it is entirely on this vocabulary that we base our reasoning.”
  - Precision in communication can start with pictorial representations such as the one shown below. In order for students to respond effectively, they have to employ precise language and to explain their thoughts either orally or in written language, possibly with the aid of symbols.



Compare the two sets shown above:

- a. Describe the ratio of shaded circles to unshaded circles in each set.
- b. How is this comparison different from describing the ratio of shaded circles to all of the circles in each set?
- c. How could you compare the ratio of shaded circles in Set A with the ratio of shaded circles in Set B?

Talking about the relationships between fractions helps you think about how these relationships can be written symbolically. If you try to use the symbols first, you may get lost in the numbers. Conversation about mathematical concepts helps foster a deeper understanding of these concepts.

retrieved February 2013 from  
[http://www.learner.org/courses/teachingmath/grades6\\_8/session\\_02/section\\_02\\_b.html](http://www.learner.org/courses/teachingmath/grades6_8/session_02/section_02_b.html)

- Clearly, students cannot practice precision if they lack opportunities to communicate inside the classroom. Some practical strategies for creating opportunities for communication within the math classroom are provided below.
  - Working in Groups: It is important for students to share with one another the strategies they use to complete mathematical tasks. This helps students make sense of the mathematics and expands their understanding and ability by explaining and listening. Designing classroom experiences around having students work in groups enables students to *ask one another questions, verbalize their thought processes, and clarify understanding as well as misunderstandings*. Retrieved February 2013 from [http://www.learner.org/courses/teachingmath/grades6\\_8/session\\_02/section\\_03\\_d.html](http://www.learner.org/courses/teachingmath/grades6_8/session_02/section_03_d.html).
  - Janet Eaton gives three recommendations to improve teaching and learning in the general education math classroom. Retrieved February 2013 from [http://www.ascd.org/ASCD/pdf/journals/ed\\_lead/el\\_198509\\_eaton.pdf](http://www.ascd.org/ASCD/pdf/journals/ed_lead/el_198509_eaton.pdf).
    - Increase the quality and quantity of communication about mathematics:
      - a) use precise mathematical language in instruction, b) have students explain their thinking and reasoning, and c) communication should focus on conceptual understanding
    - Use the social organization of the classroom to facilitate instruction: organize students into teams where they share the responsibility for each other's learning; integrate incentives for good teams
    - Teach new topics or modify existing ones and assign interesting and challenging problems
  - Communication can happen in the math classroom through gallery walks, math congress, and bansho as explained in this article, retrieved February 2013 from [http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS\\_Communication\\_Mathematics.pdf](http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Communication_Mathematics.pdf).
- On the following page is a list of suggested criteria for evaluating characteristics of “good” writing in mathematics. This list is intended for teachers to use in grading student assignments (retrieved February 2013 from <http://www3.saintmarys.edu/departments/mathematics-computer-science/department-policies/advanced-writing-requirement-mathematics/criteria-good-writing>):

1. **Accuracy:** The paper is free of mathematical errors, and the writing conforms to good practice in the use of language, notation, and symbols.
2. **Organization:** The paper is organized around a central idea. There is a logical and smooth progression of the content and a cohesive paragraph structure.
3. **Clarity:** Explanations of mathematical concepts and examples are easily understood by the intended audience. The reader can readily follow the paper's development.
4. **Insight:** The paper demonstrates originality, depth, and independent thought.
5. **Mechanics:** The paper is free of grammatical, typographical, and spelling errors. The mathematical content is formatted and referenced appropriately.



**PRACTICE #7: Look for and make use of structure**

- What does it mean to look for and make use of structure?
  - Students can look at problems and think about them in an unconventional way that demonstrates a deeper understanding of the mathematical structure – leading to a more efficient means to solving the problem.
  - Example problems from Gail Burrill:
    - Solve for  $x$ :  $3(x - 2) = 9$ 
      - ◇ Rather than approach the problem above by distributing or dividing, a student who uses structure would identify that the equation is saying 3 times something is 9 and thus the quantity in parenthesis must be 3.
    - Solve for  $x$ :  $\frac{3}{x-1} = \frac{6}{x+3}$ 
      - ◇ The “typical” approach to the above problem would be to cross multiply and solve; a student who identifies and makes use of structure sees that the left side can be multiplied by 2 to create equivalent numerators... then simply set the denominators equal and solve.
  - Many examples of a students’ ability to identify and make use of structure show up in the SAT – the example problems that follow are best approached from a structural point of view rather than a strictly procedural approach
    - Given  $a - b = 3$  and  $a^2 - b^2 = 12$ , what does  $a + b = ?$ 
      - ◇ In this example students have to identify the structure of a difference of squares and see the connection between the givens and the question.
    - If  $3x - 2y = 10$  and  $x + 3y = 6$ , what is  $4x + y$ ?
      - ◇ In this example students have to identify the structure of a system of equations... rather than solve for each variable, the most efficient approach is to look at the connection between the givens and the question and identify that the two equations can be added to produce  $4x + y$ .

- A sequence begins with the number 5. The next term is found by adding 4, and the next term is found by multiplying by -1. If this pattern of adding 4 and multiplying by -1 continues, what is the 26<sup>th</sup> term in the sequence?
  - ◇ In this example students have to identify a pattern by grouping... the sequence is 5, 9, -9, -5, 5, 9, -9, ... they can then identify the pattern repeats after groups of 4... students can then use this to see how many groups of 4 are in 26 and use the remainder to identify the correct term.

**PRACTICE #8: Look for and express regularity in repeated reasoning**

- A professional development training, retrieved February 2013 from <http://mason.gmu.edu/~jsuh4/impact/PFAEntire.pdf>, focuses on getting students to generalize processes and develop algebraic thinking at the elementary school level using Patterns, Functions, and Algebra.
- Generalizing-Promoting Actions: How Classroom Collaborations Can Support Students' Mathematical Generalizations. This article from the Journal for Research on Mathematics Education focuses on how teachers can help students to develop mathematical generalizations through collaboration (access does require NCTM membership)
- Forging Mathematical Relationships in Inquiry-Based Classrooms with Pasifika Students. This article, retrieved from <http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/100/81>, presents research on how students develop mathematical practices and competency when placed in classrooms that focus on inquiry and finding patterns.
- The Barbie Bungee activity from illuminations requires students to record data as they simulate a bungee jump with a Barbie doll. They are prompted to make generalizations about their data they find to report a general pattern.
- Example problem involving generalization and expressing regularity from Cuoco, AI, et. al. (2009). Algebra 2. *CME Project*, p. 148:
  - a) If  $a$  is some fixed number, find the general form of  $(x - a)(x^2 + ax + a^2)$ .
  - b) If  $a$  is some fixed number, find the general form of  $(x - a)(x^3 + ax^2 + a^2x + a^3)$ .
  - c) What is a general result suggested by parts (a) and (b)?
- Patterns and Functions, retrieved February 2013 from <http://www2.edc.org/mathpartners/pdfs/6-8%20Patterns%20and%20Functions.pdf>, outlines many different activities and classroom resources that are centered around pattern recognition within functions for grades 6-8.
- Learner.org provides a series of Professional Development videos that focus on helping students to find patterns and develop structures within major algebra topics. Retrieved February 2013 from <http://www.learner.org/courses/learningmath/algebra/>