# Transition to Advanced Mathematics

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Unit 2: Numbers and Integers

Unit Introduction
By the time the students have completed this unit, they will have encountered all four operations with integers as well as the commutative, associative, and distributive properties. Development of the additive identity, additive inverse, zero pair, and subtraction definition will occur in this unit. The lessons engage the students by exploring the concepts behind integers with basic manipulatives and activities and by applying real world uses for integers.

Many of the lessons in this unit engage students with tiles (specifically the unit tiles). It may be helpful to complete a short introduction to the tiles. Below is suggested introduction outline.

Getting to know the tiles
Have the students work in pairs and give each pair a set of tiles. Note: You may want to put each set of algebra tiles in a sealable sandwich bag for ease of storage. Also, this will be a good time to establish norms for using the tiles.

Have the students open the bag and categorize the items in the bag. Have the students share how they categorized the items. Students should realize there are small squares, large squares, and small rectangles. Students should also see that each tile has two colors.

Ask students an open question that invites them to share what they think these tiles could represent in mathematics class this year. Again, have the students share ideas.

Explain to students that periodically throughout the year, they will use these tiles to model various mathematical relationships and quantities.

Tell students that the length of one small square is one unit and the width is one unit. This gives an area of one square unit.

Then, ask the student to determine the width of the rectangle based on the small square unit. The students should be able to describe that the width is one unit. Now ask them to determine the length based on the small square unit. The students will find that they cannot determine the exact length. Clarify to students that because of this, we will refer to the length of the rectangle as \( x \). We will name the rectangle by its area 1 times \( x \) or \( x \).

Now ask the students to determine the length and width of the large square based either on the small square or on the rectangle. The students should realize that the length and width is the same as the rectangle’s length. Tell students that because the length and width are “\( x \)” we will call this item \( x^2 \). Described by multiplying the length times the width.
Tell students that we will use the color red to designate negative values. Ask the students what the following values would be.

One red small square  One yellow small square  One red rectangle
One green rectangle  One blue large square  One red large square

**Note:** Some teachers like to put a negative one symbol (−1) on the red side of the small square, a positive one symbol (+1) on the yellow side of the small square, −x on the red side of the rectangle, + x on the green side of the rectangle, −x² on the red side of the large square, and +x² on the blue side of the large square.

Ask students to represent the following values or expressions using the tiles. Walk around and check the representations.

- 5
- −6
- −4+3
- x + 4
- 2x + 3
- −3x + (−2)
- 2x² + 3x + 4

This should conclude the introduction students need to use the tiles effectively throughout the year. Remember to establish the norms for using the tiles. Many teachers like to create a system for storing and disseminating the tiles easily. Also having only the tiles that are needed for the exercises prevents distractions. For most of this unit the students will need only the small unit squares.
Lesson 1: Natural Numbers to Integers

In Brief
Students develop an understanding of the organizational structure and closure properties of natural numbers, whole numbers, and integers.

Objectives
• Students use graphic organizers to represent the relationships of natural numbers, whole numbers, and integers.
• Students determine if a set of numbers is closed under addition, subtraction, or multiplication.

Shaping the Lesson
• This lesson uses historical perspectives and presents a variety of scenarios to lead students to understand the need for numbers beyond whole numbers.

Instructional Strategies
• Discussion
• Guided Practice
• Outcome Sentences

Tools
• Calculator
• Dry-Erase Boards, Erasers, Markers
• Student Journal
• Chart Paper (optional)
• Blank Transparencies and Markers
• Overhead Projector or other projection device

Warm Up
Problem of the Day
Assign a group of four to a set of numbers provided in their student journals. Four different versions of numbers have been provided to give variety to the activity. *

Have each group categorize the numbers in a way that makes sense to them. At this time don't worry about defining terms, such as integer, greater than zero, etc. Just let students use their working vocabulary. Have students create a poster or transparency presentation of their categorization. You may need to suggest different methods of presentation, such as: tables, tree diagrams, Venn diagrams, etc. There will not be one appropriate approach. The goal is for students to communicate mathematical reasoning and ensure their attributes for categories are applied appropriately.

After the groups have created their posters or transparencies, have them share. Make sure to ask clarifying, probing, or guiding questions. For example, if a group says, "We put some of these numbers in a category we called less than zero", you might find it important to ask the class, "What is a term that we can use to describe numbers less than zero?" It may be helpful to guide the students to the correct vocabulary through a class discussion.

As the class develops some of the vocabulary for numbers, you may want to put the words on a word wall.

Don't worry about mastery of the vocabulary words for numbers and operations at this time. This will be developed throughout the lesson and unit.

One goal is to allow students to make links to the knowledge that they already have about numbers. This will be valuable for you, as the teacher, because you will gain insight into the misconceptions and preconceptions that your students have about numbers. You may want to share with the students that after this “Setting the Stage,” they will get to engage in a series of lessons that further develop their knowledge of numbers and operations.

* Option: It may be more engaging to give each group a set of index cards that match the given sets of numbers. This would allow students to physically move the numbers around and place them in categories. The students could even tape the numbers to poster paper to show how they categorized the numbers.

TEACHER’S NOTES

________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
Your teacher will assign you one of the following sets of numbers. Add five more numbers to your set. Organize the set of numbers into different categories. Create a display of how you categorized the numbers.

### Set A
- 3
- 2
- -5.24
- 4 1/2
- -4
- 0
- -2.333
- 2
- 4/7
- \(\sqrt{2}\)
- \(\sqrt{9}\)
- -8.2
- 3/4
- 2.3
- 24
- -14

### Set B
- \(\sqrt{25}\)
- -7
- \(\sqrt{5}\)
- \(\pi\)
- -3.75
- -4
- 1
- -8
- 4.12
- -1
- 0
- 33.3
- 1/3
- 2 1/5
- \(\sqrt{3}\)
- \(\overline{5.7575}\)

### Set C

<table>
<thead>
<tr>
<th>75%</th>
<th>(\frac{1}{3})</th>
<th>33.3</th>
<th>0.75%</th>
<th>0.33</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>(\frac{3}{4})</td>
<td>0.7</td>
<td>-30</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>33%</td>
<td>(-\frac{1}{3})</td>
<td>-33%</td>
<td>(\frac{3}{9})</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>

### Set D

<table>
<thead>
<tr>
<th>75%</th>
<th>(\sqrt{25})</th>
<th>0.252525</th>
<th>0.252525</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sqrt{5})</td>
<td>30%</td>
<td>-7</td>
<td>25</td>
</tr>
</tbody>
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Historic Review

Discuss with students that it has been in only the last few centuries that scientists and mathematicians in most cultures have worked with various types of numbers that you categorized in the “Setting the Stage.” People mainly worked with what we call the natural and whole numbers. In fact, negative numbers were not often discussed or used until after the 17th century.

Below is a suggested teaching strategy for the “Historical Review.” You may need to modify it to meet the needs of your students. You may want to complete your own research on the history of numbers. It can be a complex study and very interesting.

1. Have a student read the first two paragraphs of the Historical Review. Have partners discuss with each other the answer to this question: “Why do you think early historic cultures only worked with the natural numbers?” Give the students about one minute. Call on at least two different pairs to describe their answer. We hope students make statements such as, “It’s like counting with your fingers,” “When cultures didn’t record temperature measurements or numbers representing debt, people didn’t need negative numbers,” or “It’s just like we learned when growing up – start simple.”

As a class discuss Exercise 1 and talk about the fact that the natural numbers are an infinite set. Show them how the list shows an infinite extension by placing three periods after the last number in the list. Draw an oval on the board, name the oval, and place a list of natural numbers in the oval. This is the beginning of a Venn diagram that will represent the relationship between the different sets of numbers.

2. Talk with students briefly about the European version of the Hindu-Arabic numbers that they use today. Have them look at Brahmi numerals from which the numbers could have evolved. Ask them to find the difference and similarities between Hindu-Arabic numbers and Brahmi numbers. They may notice that connecting the two slashes for 2 and the three slashes for 3 could actually create the symbols we use today.

3. Have the students work in pairs to study the four examples of natural numbers. Then ask the pairs to think of answers for the following questions: “What do the four representations have in common? Are there attributes that only some sets share?” Students should notice that all four begin with a tallying type of representation. They may also notice that two of them continue to use the tally type method and the other two begin to use symbols to represent larger numbers.
4. Have student pairs complete Exercises 2 through 6. Have the pairs switch roles for each problem. One person will read and the other records the answer. Allow the pairs about 8 minutes to complete the exercises. Have each pair group with another pair and share their answers. Once the groups have shared and agreed, then each individual will need to record the answers in their student journal. You may find that you do not need any class sharing at this point. If you do, make it brief and keep it focused on natural or whole numbers.

Have the students share their graphic organizers for Exercise 6.
Throughout history, numbers have been used to help people record and calculate. The first set of numbers that most cultures worked with were the **natural numbers**.

Natural numbers are basically used to represent counting objects. This is why you may often hear natural numbers referred to as the **counting numbers**.

The method that you use today to represent natural numbers is the Hindu-Arabic number system. Below is a list of the European Hindu-Arabic symbols for the first 12 natural numbers.

\[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ldots\]

1. How long could you continue the list of natural numbers? **Sample Answer:** Indefinitely.

While there are different variations of the Hindu-Arabic numbers recorded over history, they probably evolved from the Brahmi numerals from India. The first nine natural numbers in Brahmi probably looked very similar to the following:

Brami (~100 AD) \[- \quad \equiv \quad + \quad \| \quad \| \quad \| \quad \| \quad \| \quad \|\]

Various cultures throughout history had different ways to represent the first nine natural numbers with symbols.

Mayan (before ~1500 AD)

Roman (~400 AD – 1500 AD)

I II III IV V VI VII VIII IX

Babylonian (~1800 BC)

Chinese (Still in use today.)
Eventually some cultures needed to represent the idea of nothing. For example, if you had 25 sheep in your flock and you traded them all for fishing nets, you would have nothing in your flock. A person could now record it as 0 sheep. Today we generally use the word “zero” instead of “nothing.”

2. List three other ways you have used zero.

   Sample responses: Zero degrees Fahrenheit, $0.00 in my savings account, I have zero siblings.

The symbol zero also helps with the modern method of using place value to represent large numbers. For example, we use a zero when we go from a number like 9 to the number 10. In this case the 0 actually means that there are zero ones and the 1 means there is one ten. Similarly, the number 400 uses zeros to represent zero ones and zero tens.

3. What do the zeros represent in the following numbers?

   a. 302 The zero represents zero tens.

   b. 4,067 The zero represents zero hundreds.

   c. 106,870 One zero represents zero ten thousands and the other represents zero ones.

4. Describe how difficult it would be if we didn’t have the number or concept of zero.

   Answers will vary. Students should recognize that we would not have place value.

Combining the number zero to the set of natural numbers makes a new set called whole numbers.

5. In your own words, describe the differences and similarities between the set of whole numbers and natural numbers.

   Answers will vary. Students should describe that they both continue indefinitely and that the only difference is the number zero.

6. To remember the difference between whole numbers and natural numbers, modify the graphic organizer below for natural numbers to include zero and a label for whole numbers. Draw your new graphic organizer on a dry-erase board, and be prepared to share your idea with the class.

   Note: This exercise is an attempt for students to create a method to remember whole numbers and natural numbers so that when you refer to these number sets later in the year they have a concept definition in mind.
**Discovery Activity**

Why did people begin to add new symbols to the whole numbers to make other sets of numbers? This activity gives a brief contrived situation that forces the students to think beyond whole numbers. The exercises are designed to simulate what it may have been like for the first people to explore negative numbers. They are not designed to be historically accurate. The goal is for students to construct the idea that integers are needed when subtracting a larger number from a smaller number. Discovery Activity 2 will formalize this understanding by investigating the closure properties.

You can have students work in small groups or pairs. Have students complete the exercises, then discuss the following question: "Why historically do you think people switched from using only whole numbers to numbers with negative signs?" Give the students one minute to brainstorm with their partners and then have a few pairs share with the class. You may want to ask a few more guiding questions to help students come to grips with the need for negative numbers or integers.

In addition to debt, what other uses can you think of for negative numbers or integers? **Students should come up with ideas like elevation or temperature.**

What would it be like if we didn't have integers or negative numbers? **Students should explain that it would be hard to talk about scenarios where going below zero makes sense, such as temperature, elevation, and debt.**

After this activity students should have a conceptual understanding of why we went from natural numbers to whole numbers and whole numbers to integers. The next activity will solidify the need for integers and the mathematical reason of the closure property with respect to addition.

If needed, you could create, or have students create, other scenarios based on elevation, temperature, or debt for students to explore.

**TEACHER’S NOTES**

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________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
In the 6th Century, negative numbers were used to represent debts in India, but negative numbers were not widely used around the world. Even in the 1700s, European mathematicians resisted the concept of negative numbers. Even prominent mathematicians of the 1800s ignored negative solutions, thinking they were meaningless. It has only been in the last two centuries that negative numbers have become widely used.

What do you imagine it may have been like for people to first explore the use of negative numbers? Complete the next two exercises to gain an understanding of the need for negative numbers.

1. In the sixth century, a farmer makes a deal to trade 7 sheep for a cart load of grain with a merchant. When the time comes to make the trade, the farmer only has 5 sheep. The merchant still makes the trade, but says, “You owe me 2 sheep.” Write an equation that represents starting with 5 sheep and subtracting 7 sheep. How does the concept of owing 2 sheep show up in the equation?

   **Sample response:** \(5 - 7 = -2\). Owing 2 sheep shows as \(-2\) in the equation.

2. Another merchant keeps track of her inventory by listing what she currently has, listing what she sells, and then records what she has left by subtracting what she sells from what she has. Complete the table below.

<table>
<thead>
<tr>
<th>Current Inventory</th>
<th>Sales</th>
<th>Left</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 Sheep</td>
<td>9 Sheep</td>
<td>(15 - 9 = 6 \text{ Sheep})</td>
</tr>
<tr>
<td>4 Nets</td>
<td>7 Nets</td>
<td>(4 - 7 = -3 \text{ Nets})</td>
</tr>
<tr>
<td>6 Bushels of Grain</td>
<td>8 Bushels of Grain</td>
<td>(6 - 8 = -2 \text{ Bushels})</td>
</tr>
</tbody>
</table>

What might it mean to the merchant to list \(-3\) nets in the third column?

**Sample response:** Listing \(-3\) nets in the third column, means that she still owes 3 nets to someone.
Guided Discussion

Now that students have constructed the need for negative numbers (integers) and a concept of how they have developed, this activity is designed to formally link the terms of whole numbers and integers as well as develop the closure property to solidify the need for integers to complete the closure property for subtraction.

To begin this discussion, first talk with students about what "closed" means to them. Have them brainstorm answers as you write them down on the board or overhead. You may want students to list these on their dry-erase boards. Students will probably list ideas such as:

- Shut or close a door or window
- Close a deal
- Something that is not open

Now have students read the “Sorry We’re Closed.” Pick a student to offer an example of adding two whole numbers and record the response on the overhead, board, or chart paper. Have that student pick another student to offer another example. Continue this strategy until you have a sufficient number of examples. A student can pass if he/she chooses, but allow only two or three passes. After you have generated the list of responses, have small groups of students brainstorm about Exercise 1 and then share responses as a class.

Have the small groups brainstorm about Exercise 2 and then have them share their ideas. The students may want to create a list of examples to convince themselves that whole numbers are closed under multiplication.

Before the small groups work on Exercise 3, again pick a student to offer an example of subtracting two whole numbers and record the response. Have that student pick another student to offer another example. Continue this strategy until you have a sufficient number of examples and a few examples that wouldn’t work, such as 5 – 9. A student can pass if he/she chooses, but allow only two or three passes. After you have generated the list of responses, have the small groups brainstorm about Exercise 3 and then share responses as a class. Students should be able to determine that whole numbers are not closed under subtraction because of examples such as 5 – 9 = –4. You may have some students suggest or discover that integers may be closed.

Before students complete Exercise 4, introduce the students to the set of integer numbers, then have the small groups prove or disprove integers closed under the three given operations. You may want to assign addition, subtraction, and multiplication to different groups. The goal would be for the groups to make a short presentation on why they believe the integers are closed or not closed under the given operation. Students could use the overhead, chart paper, board, or dry-erase boards to make their presentations.

Confirm with students that integers are closed under addition, subtraction, and multiplication. Tell students that at the end of this unit we will begin to look at the operation of division and what number sets may be closed to division.
Finally, have the groups modify the graphic organizer introduced in the Historic Review to include natural numbers, whole numbers, and integers. A sample response is below.
What does it mean for a set of numbers to be **closed under addition**? It basically means that by adding any two numbers in a set you always get another number in the set.

1. Can you think of two whole numbers that when added would give a number that is not a whole number? Explain.  
   *Answers will vary.*  
   Sample response: No matter which whole numbers I add, I always get a whole number. For example, 9 and 5 are whole numbers and their sum is a whole number. 
   \[ 9 + 5 = 14 \]

Think of the whole numbers as closed under addition by imagining all the whole numbers inside a container and no matter which two whole numbers you add together the sum would never be outside the container.

2. Are the whole numbers closed under multiplication? How do you know?  
   *Sample response: Yes. No matter which whole numbers I multiply, I always get a whole number.*

3. Are the whole numbers closed under subtraction? Explain.  
   *Sample response: No. The whole numbers are not closed under subtraction. For example, 5 – 9 = –4. –4 is not a whole number.*
   You may want to remind students that the mathematical ability to subtract a larger number from a smaller number reinforces the need for another set of numbers beyond whole numbers.

The set of **integers** includes all the whole numbers and their opposites.

\[ \{... -4, -3, -2, -1, 0, 1, 2, 3, 4...\} \]

4. Determine if the set of integers is closed under the following operations. Be prepared to share your answers.

   a. Addition **Yes**
   b. Subtraction **Yes**
   c. Multiplication **Yes**

---

**Closed**

A set of items, such as numbers, is **closed under an operation** if the operation on any members of the set always produces members of that set.
Unit 2: Numbers and Integers

Outcome Sentences

I now understand why integers __________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Integers, whole numbers, and natural numbers __________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Subtracting a large number from a smaller number _______________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Integers are important because _________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Something is closed in math ____________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Early number systems _________________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Graphic organizers are useful ___________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

An attribute of _______________________________ numbers is ____________________________
_______________________________________________________________________________________
_______________________________________________________________________________________
Lesson 1 Assessment — Teacher’s Key

1. Draw a graphic organizer that represents the organization and relationships between natural numbers, whole numbers, and integers.

   Answers will vary. A sample Venn diagram is given. Students may use other graphic organizers such as tables or tree diagrams for representations.

   ![Venn Diagram]

2. Pick a set of numbers (natural numbers, whole numbers, or integers) and describe why the set is closed or not closed under the operations of addition, subtraction, and multiplication.

   Answers will vary. The natural numbers should be closed under addition and multiplication. The whole numbers should be closed under addition and multiplication. Integers should be closed under addition, subtraction, and multiplication.
Lesson 1 Assessment

Name: ________________________________ Date: __________________

Class: ___________________ Instructor: ____________________________

1. Draw a graphic organizer that represents the organization and relationships between natural numbers, whole numbers, and integers.

2. Pick a set of numbers (natural numbers, whole numbers, or integers) and describe why the set is closed or not closed under the operations of addition, subtraction, and multiplication.

____________________________________________________________________________________
____________________________________________________________________________________
____________________________________________________________________________________
____________________________________________________________________________________
____________________________________________________________________________________
____________________________________________________________________________________
In Brief
Students will engage in activities to develop their own rules for adding integers.

Objectives
• Students use the zero pair concept to add integers.
• Students develop a rule for adding integers.
• Students add integers symbolically.
• Students add matrices with integers.
• Students confirm commutative and associative property of addition for integers.

Shaping the Lesson
• The context of tug-of-war and algebra tiles will help students construct an understanding of adding integers.

Instructional Strategies
• Discussion
• Guided Practice
• Pass It Along
• “What” Questions
•Outcome Sentences

Tools
• Tiles
• Student Journal
• “Setting the Stage” Transparency
• Overhead Projector or other projection device
• Number Line (Optional, not provided)
• Overhead Tiles
• String
• Clothespins
• Dry-Erase Boards

Warm Up
Problem of the Day
Setting the Stage

Place “Transparency 1” on the overhead and then read the following: “Have you ever heard of tug-of-war competitions? Tug of war pits two teams against each other in a pulling contest. One team aligns itself at one end of a rope while the other team aligns itself at the other end. On the start command, both teams pull against each other until one team has pulled the other team past a predetermined mark.”

You may want to have two small teams model this for the class. You could use the string supplied, but be careful the students do not pull too hard on the string and damage their hands. Use a clothespin to mark the middle of the string.

*An additional interesting fact to share with your students—In many countries there are tug of war clubs. The Tug of War International Federation holds several competitions throughout the year and has members in the following countries:

<table>
<thead>
<tr>
<th>Australia</th>
<th>Hong Kong</th>
<th>Mauritius</th>
<th>Serbia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belgium</td>
<td>India</td>
<td>Mongolia</td>
<td>Singapore</td>
</tr>
<tr>
<td>Cambodia</td>
<td>Iran</td>
<td>Morocco</td>
<td>South Africa</td>
</tr>
<tr>
<td>Cameroon</td>
<td>Ireland</td>
<td>Namibia</td>
<td>Sri Lanka</td>
</tr>
<tr>
<td>Canada</td>
<td>Israel</td>
<td>Nepal</td>
<td>Sweden</td>
</tr>
<tr>
<td>Channel Islands</td>
<td>Italy</td>
<td>Netherlands</td>
<td>Switzerland</td>
</tr>
<tr>
<td>China</td>
<td>Japan</td>
<td>Nigeria</td>
<td>Turkey</td>
</tr>
<tr>
<td>Chinese Taipei</td>
<td>Kenya</td>
<td>Northern Ireland</td>
<td>Ukraine</td>
</tr>
<tr>
<td>Czech Republic</td>
<td>Korea</td>
<td>Pakistan</td>
<td>Vietnam</td>
</tr>
<tr>
<td>England</td>
<td>Latvia</td>
<td>Philippines</td>
<td>Wales</td>
</tr>
<tr>
<td>France</td>
<td>Lithuania</td>
<td>Poland</td>
<td>Zambia</td>
</tr>
<tr>
<td>Germany</td>
<td>Macau</td>
<td>Russia</td>
<td>USA</td>
</tr>
<tr>
<td>Greece</td>
<td>Malta</td>
<td>Scotland</td>
<td></td>
</tr>
</tbody>
</table>

Place “Transparency 2” on the overhead. Tell students, “We will use the tug-of-war concept to help us understand integers. We will assume that each person is exactly the same size and pulls with exactly the same force.” With these conditions, have the students answer the questions on the transparency. The goal is for students to understand that there is no movement. We will eventually translate this concept to zero pairs later in the lesson.

Place “Transparency 3” on the overhead. As a class, work through the four different situations. The goal is for students to gain an intuitive feeling for which side has the advantage. **Note:** You may want to show students how to cancel one person on the left with one person on the right.
TUG OF WAR

On the “start” command, both teams pull against each other until one team has pulled the flag in the middle past a predetermined mark.

Interesting Facts
• From 1904 to 1930, Tug of War was an Olympic event.
• If you lived in England, you could belong to the England Tug of War Association and compete in the European Tug of War Championship or World Championship.
If there is only one person on each end of the rope and each person pulls with exactly the same force, what would happen?

If there are two people on each end of the rope and each person pulls with exactly the same force, what would happen?
**Tug of War**

For each situation, determine which team would have the advantage. The negatives pull to the left and the positives pull to the right.
**Discovery Activity**

You will need the red and yellow unit tiles supplied in the materials kit for the overhead and for the students.

Talk with students about using a red tile to represent a “negative” team member and a yellow tile to represent a “positive” team member. Show students how to represent different integer values using the overhead tiles. For example, you may want to show that three yellow tiles represent the value of 3, while three red tiles represent the value of −3.

Have students work in groups of four to complete Exercises 1 and 2.

After students have completed Exercises 1 and 2 and shared their results with each other, discuss the concept of the red and the yellow tiles representing two opposite people pulling in a tug-of-war contest. With both representations, the idea is "nothing happens." Guide students to the idea that together one red tile and one yellow tile represents zero, just like a −1 and a +1 together equal zero. These opposite pairs are referred to as a "zero pair." It will help students use tiles if they begin to think of "zero pair" when they see one red and one yellow tile together.

At this time you will want to establish a definition of "zero pair." Have students help you create a definition in their own words. A sample definition is given below. You may want to place "zero pair" on a “Word Wall” and/or have students record it in their Vocabulary Organizer.

**Zero Pair:** A sum of positive one and negative one.

For example, \( 1 + (–1) = 0 \)

You may find value in explaining how this idea expands beyond units. For example, these are also zero pairs: \( 4 + (–4) = 0 \) and \( x + (–x) = 0 \)

**Note:** You may need to explain your preference for writing the subtraction or addition of a negative number. For example, you may prefer \( 4 + (–3) \) instead of \( 4 + (–3) \).

Transition students to the situation with Sheena and Kendra by placing the transparency titled “My Dollars or Yours” on the overhead. Think aloud as you guide students through this scenario. Use tiles to keep track of how much money Kendra borrowed and how much she lent. You may want to write the corresponding math symbols. An example of how to use the tiles is shown below. You may want to ask students what could represent borrowing and lending $1.00.

*A red tile could represent each dollar that Kendra borrows, because Sheena is missing that dollar. Let a yellow tile represent each dollar that Kendra lends, because Sheena gets that dollar back.*

- **Kendra borrowed $3.00 for pizza.**
  - \( \begin{array}{c}
    \text{red tiles} \\
    3
  \end{array} \) 
  - \( 3 \)

- **Kendra lent $2.00 for a soda.**
  - \( \begin{array}{c}
    \text{yellow tiles} \\
    2
  \end{array} \) 
  - \( –3 + 2 \)

- **Kendra lent $5.00 for popcorn.**
  - \( \begin{array}{c}
    \text{yellow tiles} \\
    5
  \end{array} \) 
  - \( –3 + 2 + 5 \)

- **Kendra borrowed $1.00 for cookies at lunch.**
  - \( \begin{array}{c}
    \text{red tile} \\
    1
  \end{array} \) 
  - \( –3 + 2 + 5 + (–1) \)
Kendra borrowed $3.00 for class.  \[-3 + 2 + 5 + (-1) + (-3)\]

We can regroup the red and yellow tiles to create zero pairs. Just like 1 + (–1) is a zero pair, borrowing $1.00 and lending $1.00 is also a zero pair.

\[-3 + 2 + 5 + (-1) + (-3) = 0\]

Kendra owes Sheena $0.00 and Sheena owes Kendra $0.00.

You may want to point out that the story has reversibility by looking at the scenario from Sheena’s point of view. The color of the tiles would be reversed. You may also want to have a student create a scenario in which the money borrowed and lent does not come out to be the same.

After you have modeled this situation, have the students parallel you as you model an example of adding zero pairs to a number. While you show an example, have the students model a different example. Each student will more than likely have a different example. The goal is for students to understand that they can add any number of zero pairs to a number without affecting the value of that number. An example is displayed below that shows the teacher example, along with the steps, and with a student example. You may want to give the steps both orally and on the overhead (or board). Have students represent the steps with algebra tiles (or drawings of tiles) and with symbols simultaneously.

Have the students hold up their dry-erase boards for you to make a quick assessment.

<table>
<thead>
<tr>
<th>Teacher Model</th>
<th>Steps</th>
<th>Student Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>[-3]</td>
<td>Pick an integer</td>
<td>2</td>
</tr>
<tr>
<td>[-3 + 1 + (-1)]</td>
<td>Add one zero pair</td>
<td>2 + 1 + (–1)</td>
</tr>
<tr>
<td>[-3 + 1 + (-1) + 1] + 1 + (–1)</td>
<td>Add a few more zero pairs</td>
<td>2 + 1 + (–1) + 1 + (–1) + 1 + (–1)</td>
</tr>
<tr>
<td>[-3 + 4 + (-4)]</td>
<td>Collect all the zero pairs and record as one set.</td>
<td>2 + 5 + (–5)</td>
</tr>
<tr>
<td>[-3]</td>
<td>Determine value.</td>
<td>2</td>
</tr>
</tbody>
</table>

Complete another example as needed.

Now, have students work in groups, or with partners, to complete Exercises 3 through 5. The goal is for students to create a rule, or conjecture, for each type of addition of integers that can occur. Because this is one of the main goals of the activity, you will want to have some of the groups share their rules. It may help to have the groups come to the overhead and, while sharing their rules, show how it works with
tiles. You could test the rules with the extra examples given below. You could also use the parallel method described previously to work through these examples, while students work through their own. A good rule should include something about the addition and about the sign. See sample answers for Exercises 3 and 4.

a.  $5 + 9 = 14$

b.  $-7 + (-4) = -11$

c.  $-9 + 5 = -4$

d.  $8 + (-5) = 3$

Optional Manipulative for Adding Integers
Some teachers and students have used number lines to develop a conceptual understanding of adding integers. While we have not supplied an activity based on number lines in this lesson, we have created two examples that are shown below. Depending upon the needs of your students, you may want to use number lines as an additional activity. Remember, the goal of the activity is for students to create conjectures for adding integers.

Example 1:  $-4 + (-3) = -7$. Start at zero and go 4 units to the left, go another 3 units to the left and the result is the same as going 7 units to the left.

Example 2:  Start at zero and go 4 units to the left, go 3 units to the right and the result is the same as going 1 unit to the left.

Some teachers place large number lines with tape on the floor, or wall, so students can model adding integers by walking. For example, to walk $-4 + (-3) = -7$, a student would start at “zero” and walk to the left four units, pause, then walk three more units to the left. The result would be the same as walking seven units to the left arriving at negative seven. To walk $-4 + 3 = 1$, a student would start at “zero” and walk to the left four units, pause, then walk to the right three units. The result would be the same as walking to the left one unit arriving at negative one.
For this part of the Discovery Activity, you will need a set of tiles. Have one person from your group get the tiles.

Imagine you have people pulling as before. Only this time, use yellow tiles to represent a person pulling to the right and red tiles to represent a person pulling to the left. In other words, yellow tiles represent positive and red tiles represent negative. Use your tiles to model and then sketch a diagram for each situation.

*Remember, it will be very important to wrap up this activity by having students respond to part “b” of both exercises below. This is where students can connect their new understanding to the skills needed to add integers. This will also give students the opportunity to think about their thinking. Remember to try the rules the students give with an example to see how well their rule works.*

1. Six people pull to the right and two people pull to the left.
   a. Sketch a diagram below.
      
      *Sketches will vary.*

   b. Which team will win? What advantage will the winning team have? Justify your answer. *Sample answer: The positives will win. The two people on the left cancel two of the six people on the right, which leaves four extra people pulling to the right.*

2. Five people pull to the left and three people pull to the right.
   a. Sketch a diagram below.
      
      *Sketches will vary.*

   b. Which team will win? What advantage will the winning team have? Justify your answer. *Sample answer: The negatives will win. The three people on the right cancel three of the five people on the left, which leaves two extra people pulling to the left.*
3. Use tiles to model the addition problems below.
   a. $7 + 2$
      \[ 9 \]
   b. $6 + 5$
      \[ 11 \]
   c. $-7 + (-2)$
      \[ -9 \]
   d. $-6 + (-5)$
      \[ -11 \]
   e. What do you notice when adding numbers with the same sign? *Sample response:* 
      *To add integers with the same sign, add the numbers and keep the sign.*

4. Use tiles to model the following addition problems.
   a. $-7 + 2$
      \[ -5 \]
   b. $-6 + 5$
      \[ -1 \]
   c. $7 + (-2)$
      \[ 5 \]
   d. $6 + (-5)$
      \[ 1 \]
   e. What did you notice about these problems? *Sample response:* 
      *To add integers with different signs, subtract the numbers and keep the sign of the larger number.*

5. Write a rule or conjecture about adding integers. *Answers will vary.*
Lesson 2: Discovery Activity Transparency Master

About a week ago, Sheena complained to Kendra that Kendra always owed her money. Sheena asked her to share all of the occasions Kendra borrowed and lent money in the past month. Her list is shown below.

1. Kendra borrowed $3.00 from Sheena for pizza.
2. Kendra lent $2.00 to Sheena for a soda.
3. Kendra lent $5.00 to Sheena for popcorn.
4. Kendra borrowed $1.00 from Sheena for cookies at lunch.
5. Kendra borrowed $3.00 from Sheena for class.

Does Kendra borrow more money than she lends?

How much money does Kendra owe Sheena? How much money does Sheena owe Kendra?
Math at Work and Symbolize It

Have groups work through the “Math at Work.” You may need to guide students through the first page of the “Math at Work” because matrices may be new to students. Encourage the students to use algebra tiles as needed.

Depending on time, you may want students to do the “Symbolize It” for homework. If needed, have students continue practicing with the tiles with “Symbolize It.”

Note: Because the commutative property of addition, associative property for addition, and the additive identity are addressed in this “Symbolize It,” you may want to make sure to have a short discussion on these after students have completed this “Symbolize It.”

Give students at least 10 minutes to complete their learning log and thinking question.

You may want students to experiment with 2 x 2 matrices and the commutative and associative properties of addition.

TEACHER’S NOTES

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________
Teacher Note: Work through this page with the class before students complete the exercises.

Matrix Addition

A matrix can be used to organize and record data. For example, a clothing company may use a positive number to record the number of items it has in stock and a negative number for the number of items that are on order. The company may also use a positive number to record the number of items added to stock and a negative number for the number of items subtracted from stock. Suppose the information below is a list of the company’s in-stock items for January 1st.

<table>
<thead>
<tr>
<th>January 1st</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Small Jean Jacket:</td>
<td>30</td>
</tr>
<tr>
<td>Medium Jean Jacket:</td>
<td>11</td>
</tr>
<tr>
<td>Large Jean Jacket:</td>
<td>12</td>
</tr>
<tr>
<td>Small Winter Parka:</td>
<td>22</td>
</tr>
<tr>
<td>Medium Winter Parka:</td>
<td>5</td>
</tr>
<tr>
<td>Large Winter Parka:</td>
<td>10</td>
</tr>
<tr>
<td>Small Fleece Coat:</td>
<td>-10</td>
</tr>
<tr>
<td>Medium Fleece Coat:</td>
<td>9</td>
</tr>
<tr>
<td>Large Fleece Coat:</td>
<td>7</td>
</tr>
</tbody>
</table>

Notice how each position in the matrix relates to a specific type of clothing.

Discuss as a group what the following matrix could represent.

Discuss with students that this matrix could represent the current stock, or the matrix could represent a change in stock.

1. If the matrix below represents the transactions that occurred during January, create a matrix that would show the number of items in stock for February 1st? Hint: Add each item from the original matrix with each item in this matrix.

Some students may include the names of the columns and rows.

Matrices may be added if they have the same dimensions. Find the sum of numbers in the same location, then place that number in the corresponding location of the new matrix.
2. What matrix could you add to February 1st’s matrix so that all items in stock for March 1st would total 50? Some students may include the names of the columns and rows.

\[
\begin{bmatrix}
-25 & 18 & 39 \\
30 & 38 & 56 \\
46 & 18 & 68
\end{bmatrix}
\]

3. Create your own matrix to represent a real-world application. Add a second matrix to it and explain what the result represents. Answers will vary.
Unit 2: Numbers and Integers

Evaluate each expression below.

1. \(-3 + 5 = 2\)  
2. \(6 + (-7) = -1\)  
3. \(-2 + (-5) = -7\)  
4. \(3 + (-6) = -3\)  
5. \(9 + (-4) = 5\)  
6. \(1 + (-6) = -5\)  
7. \(-2 + 8 = 6\)  
8. \(-8 + (-4) = -12\)

9. List three examples of zero pairs. **Answers will vary.**

Any two numbers that sum to zero can be called a “zero pair.” Another name for zero is the **additive identity**; because, if you add zero to any number you still have that number. Use the symbol for the additive identity to complete the expressions below.

10. \(5 + 0 = 5\)
11. \(-7 + 0 = -7\)
12. \(n + 0 = n\)

An expression can be used to model a real-world problem. For example, the expression \(15 + (-20)\) could represent the temperature in the morning starting at 15 degrees followed by a drop of 20 degrees by evening. Write a word problem to model each expression below.

13. \(2 + (-5)\)  
   **Answers will vary. Sample response:** You walked up 2 flights of stairs and then walked down 5 flights of stairs.

14. \(-90 + (-60)\)  
   **Answers will vary. Sample response:** You owe $90.00 on your credit card and you recently charged $60.00 for a new bike.
15. Liz told her teacher she could add two whole numbers in any order and always get the same sum. She came to the overhead and showed the following two examples.

\[ 12 + 14 = 26 \text{ and } 14 + 12 = 26 \]
\[ 32 + 8 = 40 \text{ and } 8 + 32 = 40 \]

Liz’s teacher said, “That’s true. We call that idea the **commutative property**.” She asked Liz if this commutative property works for integers too. Liz wasn’t sure. If you were Liz’s partner, describe how you might help her understand whether the commutative property is true for integers or not.

*Answers will vary.*

16. While Liz was doing her homework after school, she told her big sister about the commutative property. Her big sister said that was one of the first properties she remembered learning too. She also said that she learned an even more complicated property called the **associative property**. She said, “The associative property actually works with three numbers that you want to add together. It means you can add the first two numbers or the second two numbers first and you will get the same answer. It’s like three friends meeting at the mall. It doesn’t matter which two friends get together first. Eventually, all three friends will be together.” Then she drew an example for Liz, but it was with letters. How could Liz’s big sister use numbers to help Liz understand the associative property for integers better.

\[ (a + b) + c = a + (b + c) \]

*Answers will vary.*
Unit 2: Numbers and Integers

Outcome Sentences

Today I learned that ______________________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

I was surprised that ______________________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

I feel like I really understand _____________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

I am still confused about _________________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Thinking Question

Given the temperatures below in Fahrenheit, explain how you could use adding integers to
determine the average temperature in Fairbanks, Alaska for the year. *Answers will vary.*

<table>
<thead>
<tr>
<th>Month</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>−10°F</td>
</tr>
<tr>
<td>February</td>
<td>−4°F</td>
</tr>
<tr>
<td>March</td>
<td>11°F</td>
</tr>
<tr>
<td>April</td>
<td>31°F</td>
</tr>
<tr>
<td>May</td>
<td>48°F</td>
</tr>
<tr>
<td>June</td>
<td>60°F</td>
</tr>
<tr>
<td>July</td>
<td>64°F</td>
</tr>
<tr>
<td>August</td>
<td>57°F</td>
</tr>
<tr>
<td>September</td>
<td>46°F</td>
</tr>
<tr>
<td>October</td>
<td>25°F</td>
</tr>
<tr>
<td>November</td>
<td>4°F</td>
</tr>
<tr>
<td>December</td>
<td>−7°F</td>
</tr>
</tbody>
</table>
In Brief

Students investigate subtraction of integers using two color tiles and discover that subtracting an integer is the same as adding the opposite.

Objectives

• Students model subtracting integers with tiles.
• Students develop algorithms for subtracting integers.
• Students apply the operation of subtracting and adding integers to a real-world application.

Shaping the Lesson

• Students relate subtracting from their life and tiles to the symbolic representation of subtracting.

Instructional Strategies

• Lecture
• Discussion
• Guided Practice
• Motivating Question
• Outcome Sentences

Tools

• Tiles
• Student Journal
• Dry-Erase Boards
• Overhead Algebra Tiles
• Blank Transparency
• “Setting the Stage” Transparency
• Overhead Projector or other projection device

Warm Up

Problem of the Day
**Setting the Stage**

Have students brainstorm words that mean subtract. They can record the words on the dry-erase boards. Describe to students that these words can be used for both mathematical ideas as well as ideas in their life. For example, share with students something positive in your life that was taken away and something taken away that was negative. You may want to share something like a sibling being subtracted from your home because he/she goes off to college. Another example could be an illness going away. In the first case, a positive thing was taken away which was negative overall. In the second case, a negative thing was taken away which was positive overall. Ask students if they can think of when something positive or negative was taken out of their life.

Possible guiding questions include:

- When do you subtract from your life?
- Are there any negative influences you have subtracted from your life?
- Have you subtracted a bad thing away from your life?
- Have you subtracted a good thing away from your life?

One goal is for students to associate subtraction with "take away." Another goal is for students to make links of subtracting a positive and subtracting a negative thing from their life.

You may want to review addition rules with a few examples like the following.

\[
3 + 4 \quad 3 + (-4) \quad -3 + 4 \quad -3 + (-4)
\]
Discovery Activity 1

Complete a quick review of adding integers with tiles. Use overhead tiles and have students come to the overhead to model. Students who can both symbolically add integers and conceptually understand the addition of integers will have a better foundation for subtracting integers. As the teacher, you can highly influence students’ acceptance of working with tiles by the enthusiasm you bring to using the tiles. The following four examples are good to use to review adding integers with tiles.

Guide the students through Examples 1 and 2. Have students work with a partner on Exercises 1 through 4. Suggestions are given below for grouping students into pairs. Each set of partners should have one set of tiles. Make sure the students can model each exercise with tiles. Have different individual students model the exercises at the overhead with the overhead tiles supplied in the materials kit. Be sure that each student is participating equally. The directions are in the student journal.

Guide the students through Example 3 as they try the "On Your Own" example before they complete Exercises 5 through 8. You could use the parallel modeling technique used in Lesson 2’s “Discovery Activity.” You may need to work with individual students, groups, or the entire class to help students conceptually understand how to subtract integers. The concept is to add zero pairs until there are enough correctly signed tiles to actually take away. Students may become frustrated with using tiles to complete these subtractions. They may want to try to do it only symbolically. Try to help these students do it both ways reminding them that the goal is not only to develop the skill to subtract integers but also understand why integer subtraction works.

Note: Parentheses are generally used in this lesson to distinguish a negative sign. For example, we will generally write 3 – (–4) instead of 3 – –4. You may want to have a brief discussion with students about this symbol usage.

Grouping Strategy for Partners

- Have students line up along the wall according to how comfortable they are with subtraction. Have the line fold itself in half to make pairs.
- Have students record their favorite food (or song, color, etc.) on an index card and try to find another student with the same word on his or her card.
Optional Activity

Some teachers and students are familiar with using number lines or other manipulatives to model integer addition and subtraction. We have not supplied an activity based on number lines or other manipulatives. We would like to remind teachers who may want to supplement with other manipulatives that the goal of the activity is for students to develop a conceptual understanding of subtracting integers that leads to the creation of rules for subtracting integers. Two examples of subtracting integers with a number line are shown below.

**Example 1**: $-4 - (-3) = -1$. Start at zero and go 4 units to the left, go the opposite of another 3 units to left (which is the same as going 3 units to the right) and the result is the same as going 1 unit to the left.

![Number line example 1](image1.png)

**Example 2**: $-4 - 3 = -7$. Start at zero and go 4 units to the left, go another 3 units to the left and the result is the same as going 7 units to the left.

![Number line example 2](image2.png)
Discovery Activity 1

Lesson 3

Unit 2: Numbers and Integers

Taking Away to Subtract

Example 1:
Think of 5 – 3 as taking 3 away from 5.

Begin with 5 positive tiles.

\[
\begin{array}{cccccc}
+ & + & + & + & + \\
\end{array}
\]

Take 3 positive tiles away.

\[
\begin{array}{cccc}
+ & + & + & - \\
\end{array}
\]

There are 2 positive tiles left, so 5 – 3 = 2.

Example 2:
Think of –6 – (–2) as –6 minus –2 or as taking –2 away from –6.

Begin with –6, represented with 6 negative tiles.

\[
\begin{array}{cccccc}
- & - & - & - & - & - \\
\end{array}
\]

Take –2 away. In other words, take 2 negative tiles away.

\[
\begin{array}{cccc}
- & - & - & - & - & - \\
\end{array}
\]

There are 4 negative tiles left representing –4, so –6 – (–2) = –4.

Exercises: Use tiles to model and simplify each expression below.

1. \[6 - 2 = 4\]

2. \[4 - 1 = 3\]

Check for use of tiles.

3. \[–4 – (–2) = –2\]

4. \[–5 – (–1) = –4\]

Check for use of tiles.
Adding to Subtract

Example 3:
Think of $2 - 5$ as taking 5 away from 2.

Begin with 2 positive tiles.

Begin with 2 positive tiles.

By using the previous method you would need to take away 5 positive tiles.

But we have only 2 positive tiles!! How can we take away 5 positive tiles?

This is where our work with ZERO PAIRS will help us. Adding zero to a number doesn’t change the value of a number, we can add zero as many times as we want.

We can model the value of 2 with two positive tiles and three zero pairs.

We can also think of 2 as $2 + 3 + (-3)$ which is $5 + (-3)$.

By adding three zero pairs we now have five positive tiles that we can take away.

There are 3 negative tiles left, which is the same as $-3$, so $2 - 5 = -3$.

On Your Own Example
Use tiles to solve the exercise below.

Model how to simplify the expression $-4 - 3$ with tiles. Use zero pairs to help you when needed. Draw your answer below. Then check your answer on the next page.

Answers will vary.
Unit 2: Numbers and Integers

Answer to On Your Own Problem

Begin with \(-4\) or 4 negative tiles.

\[
\begin{array}{cccc}
- & - & - & - \\
\end{array}
\]

We want to take 3 positive tiles away, so we need to add 3 zero pairs.

\[
\begin{array}{cccc}
\end{array}
\]

So, now we can take 3 positive tiles away.

\[
\begin{array}{cccc}
\end{array}
\]

There are 7 negative tiles left which has a value of \(-7\).

You might be thinking, “Why couldn’t we take away three red tiles?” It’s because each red tile represents \(-1\), not 1.

Exercises: Use tiles to model and simplify each expression below.

5. \(4 - 6 = -2\)

6. \(-1 - 6 = -7\)

Check for use of tiles.

7. \(-1 - (-4) = 3\)

8. \(3 - (-2) = 5\)

Check for use of tiles.
Discovery Activity 2

With the class, work through Exercises 1 through 6. Guide students when necessary but solicit as much information as you can from students to complete the exercises. Make sure to have students come to the overhead to model with tiles. After you have completed Exercises 1 through 6, talk with the students about opposites (additive inverses). Have them give you a few examples. Add the word "opposite" or "additive inverse" to your word wall. Tell students that they will use opposites to complete integer subtraction.

Have the students write five examples of adding opposites. Then ask them some questions that get them to think about their thinking with regard to opposites. Sample questions are given below.

- Does every number have an opposite?
- Does zero have an opposite?
- Is there an opposite to adding?
- Does every operation have an opposite?
- How do zero pairs represent opposites?

Pass out calculators and model how to complete a few integer subtractions and additions with the calculator. Some examples could be: –5 + (–9) or 15 – (–7). Have students work in pairs to complete Exercise 7.

After pairs have completed Exercise 7, make two columns on the board, one column for subtraction and one column for addition. Have students help you write down all the pairs of related expressions that have the same numbers but subtraction has been changed to addition of the opposite. See chart below. Show students that all the pairs have the same solutions.

<table>
<thead>
<tr>
<th>Subtraction</th>
<th>Addition</th>
</tr>
</thead>
<tbody>
<tr>
<td>–1 – 4 = –5</td>
<td>–1 + (–4) = –5</td>
</tr>
<tr>
<td>2 – 8 = –6</td>
<td>2 + (–8) = –6</td>
</tr>
<tr>
<td>–3 –(–9) = 6</td>
<td>–3 + 9 = 6</td>
</tr>
<tr>
<td>7 – 5 = 2</td>
<td>7 + (–5) = 2</td>
</tr>
<tr>
<td>9 – (–5) = 14</td>
<td>9 + 5 = 14</td>
</tr>
</tbody>
</table>

For Exercise 8, have the students study the two columns and then write a rule that relates addition and subtraction of integers. Have a few students present their rule. Guide the class with questions until the class has created a rule that basically represents the concept that, "subtracting is the same as adding the opposite (adding the additive inverse)." You may want to record the final rule on poster paper and post it in the room.
End the activity by reading the following scenario. You could also write the scenario on the board or overhead. Have individual students write a response and then share the response with a partner. This could be the exit pass for the day.

Scenario: Sabrina, a sixth-grade student, and her teacher had the following discussion:

Teacher: "When you subtract, will you always get a smaller number?"

Sabrina: "Yes."

Teacher: "Why?"

Sabrina: "Because when you take something away you have less."

Have students explain why they believe Sabrina is correct or incorrect. Remember to have individuals write an answer first then share with a partner before a few students share with the whole class. **Sample response: Sabrina is incorrect because subtracting a negative number is the same thing as adding a positive number which increases the value.**

Some thoughts you would like students to understand at the end of the lesson are:

- If you take away in the whole number system, the values become smaller.
- If you take away in the integer system, the values may become smaller or may become larger.
1. Write a mathematical expression in the box for the verbal sentence below.

   Subtract 5 from 3

   Sample response: $3 - 5$

2. Simplify the expression from Exercise 1 using tiles. Write the answer below.

   Sample response: $3 - 5 = -2$

3. Describe in two sentences how you determined the answer using tiles.

   Sample response: First, I added two zero pair tiles to three positive tiles, then I subtracted the five positive tiles. Two negative tiles were left over.

4. Write a mathematical expression for the following verbal sentence in the box.

   Add 3 and -5

   Sample response: $3 + (-5)$

5. Simplify the expression from Exercise 4 using tiles. Write the answer below.

   Sample response: $3 + (-5) = -2$

6. Describe in two sentences how you determined the answer using tiles.

   Sample response: I placed 3 positive tiles down and then placed 5 negative tiles, then I removed the 3 zero pairs. There were 2 negative tiles left over.

The **opposite** of a number is the number that when added to the original number gives zero. For example, the opposite of 8 is $-8$, because $8 + (-8) = 0$. The opposite of a number is also called the **additive inverse**.
7. Use your calculator to simplify the following expressions. Be sure that you enter the correct sign for the negative symbol and the correct sign for the subtraction symbol. Write the answers in the spaces provided.

8. Did you notice a pattern in the problems above? Yes.

   a. Create three examples that follow the pattern.
      \[
      4 - 7 = 4 + (-7) = -3 \\
      6 - (-2) = 6 + 2 = 8 \\
      10 - 15 = 10 + (-15) = -5
      \]

   b. What do you observe about your result? Subtracting is the same as adding the opposite.

   c. Write your rule. To subtract add the opposite.

   d. Does your rule always work? Yes.

9. Use your rule to complete the following integer subtraction.

   \[-2 - 4 = -6\]
1. Simplify each expression below using tiles or drawing a sketch of tiles.
   a. \(-2 - 6 = -8\)  
   b. \(4 - (-1) = 5\)
   c. \(-7 - (-3) = -4\)  
   d. \(3 - 5 = -2\)

2. Simplify each expression below by changing subtraction to an equivalent addition.
   a. \(6 - (-4) = 10\)  
   b. \(-7 - 2 = -9\)
   c. \(1 - 9 = -8\)  
   d. \(-5 - 9 = -14\)

3. Determine the missing value in each equation below that makes the equation true.
   a. \(-2 - (-6) + \, \text{____} \, -5 = -1\)  
   b. \(3 - 7 + 8 - \, \text{____} \, -8 = 12\)
   c. \(\, \text{____} \, 8 - 8 + 1 - 5 = -4\)

4. Solve the equations below for \(x\). You may need to use a guess and check strategy.
   a. \(x - 7 = -10\)  
   b. \(x - (-6) = 8\)  
   c. \(x - (-6) = -5\)

5. We know that there is a commutative property for adding integers. For example,
   \(8 + (-9) = -9 + 8\)

Determine if there is a commutative property for subtracting integers.

*Answers will vary. Sample response: There is not a commutative property for subtracting integers because of a counter example.*

\[8 - (-9) \neq -9 - 8\]

\[17 \neq -17\]
1. In the sport of golf, each hole has a predetermined number of strokes for the player to get the ball in the hole. This is called **par**. Scoring “on par” means getting the ball in the hole using the exact number of strokes given. Scoring “under par” means using fewer strokes and scoring “over par” means using more strokes. Sample results for the first nine holes of a round of golf are shown below.

<table>
<thead>
<tr>
<th>Hole</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strokes</td>
<td>3 over par</td>
<td>2 under par</td>
<td>1 under par</td>
<td>3 under par</td>
<td>2 over par</td>
<td>4 over par</td>
<td>5 under par</td>
<td>1 under par</td>
<td>2 under par</td>
</tr>
</tbody>
</table>

a. How has the person done overall? **5 under par**

b. To be on par for the entire 18 holes, how many under or over par should the player be for the next set of nine holes? **The player can be 5 over par.**

c. Create your own game where three holes are under par and four holes are over par and the person is 3 under par at the 9th hole. **Answers will vary.**

2. You will recall that matrices can be used to record data. Study the subtraction of matrices below and write a description of what it might represent in the real world.

\[
\begin{bmatrix}
5 & 9 & 12 \\
8 & 2 & -3
\end{bmatrix}
- 
\begin{bmatrix}
2 & 10 & 8 \\
-7 & 2 & -4
\end{bmatrix}
= 
\begin{bmatrix}
3 & -1 & 4 \\
15 & 0 & 1
\end{bmatrix}
\]

**Answers will vary. Sample response:** The first matrix represents the items in stock at a local market. The subtraction of the second matrix represents a change in stock. The last matrix represents what is in stock after the change.
3. Reach for the Stars

Hiking is a popular outdoor sport. There are many trails that follow mountain ranges allowing hikers to reach the top of a high mountain by foot. While the goal of these trails is to reach the top, the trails will go up and down as they follow the natural shape of the mountain.

You’ve decided to take a hike and have been recording your ascent and descent each hour. Your record is as follows: ascended 300 feet, descended 50 feet, ascended 200 feet, descended 400 feet, ascended 100 feet, ascended 200 feet, descended 150 feet, and finally ascended 350 feet.

If the mountain is 700 feet tall from the base to the top, how many more feet do you need to ascend to reach the summit?

\[
300 + (-50) + 200 + (-400) + 100 + 200 + (-150) + 350 = 550
\]

You need to ascend another 150 feet to reach 700 feet.

Draw a line graph of the hike on the graph below.
4. Wins and Losses
In order to have a winning record, teams must have won more games than they lost. Teams are ranked according to how many more games they have won than lost. Given the numbers of wins and losses for each team below, rank each team by record.

<table>
<thead>
<tr>
<th>Team Name</th>
<th>Number of Wins</th>
<th>Number of Losses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Springdale Spartans</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>Kenwood Kickers</td>
<td>14</td>
<td>18</td>
</tr>
<tr>
<td>Crafton Cubs</td>
<td>19</td>
<td>14</td>
</tr>
<tr>
<td>Turnersville Tigers</td>
<td>19</td>
<td>13</td>
</tr>
<tr>
<td>Edgewood Eagles</td>
<td>13</td>
<td>18</td>
</tr>
</tbody>
</table>

**Ranking by Wins minus Losses**
**Turnersville 6, Crafton 5, Springdale -2, Kenwood -4, Edgewood -5**

a. If there are five more games to play, which teams could be ranked number one? 
*Crafton Cubs or Turnersville Tigers.*

5. Running a Business
Valerie is starting a business as a wedding planner. She’s been recording her earnings and expenses over the last month. At the beginning of the month, she spent $500.00 on business cards and advertisements. She spent $50.00 on a new phone line, and she spent $150.00 on a website. She worked two weddings, earning $300.00 each. How much money did she have at the end of the month?

$$(-500) + (-50) + (-150) + 300 + 300 = -100$$

*Valerie owes $100.00*
**Outcome Sentences**

Subtracting integers is finally clear to me because taking away tiles made me think about

_____________________________________________________________________________________

_____________________________________________________________________________________

I still don't understand __________________________________________________________________

_____________________________________________________________________________________

_____________________________________________________________________________________

Using the algebra tiles helped me __________________________________________________________________

_____________________________________________________________________________________

_____________________________________________________________________________________

Today, I learned _________________________________________________________________________

_____________________________________________________________________________________

_____________________________________________________________________________________

One thing I recently subtracted from my life was __________________________________________________________________

_____________________________________________________________________________________

_____________________________________________________________________________________

Adding opposites _________________________________________________________________________

_____________________________________________________________________________________

_____________________________________________________________________________________

Integers are needed to ___________________________________________________________________

_____________________________________________________________________________________

_____________________________________________________________________________________
Lesson 4: Multiplying Integers

In Brief
Students review the rules for multiplying integers and use manipulatives to represent integer multiplication.

Objectives
• Students represent integer multiplication with tiles.
• Students solve problems with integer multiplication.
• Students simplify expressions that contain integers.

Shaping the Lesson
• Various contexts are used to help students understand and remember the rules for multiplying integers.

Instructional Strategies
• Discussion
• Lecture
• Guided Practice
• Four Corners
• Outcome Sentences

Tools
• Tiles
• Deck of Playing Cards
• Calculators
• “Setting the Stage” Transparency
• Overhead projector or other projection device

Warm Up
Problem of the Day
Setting of the Stage

In this lesson, students will use tiles with a repeated addition method to develop rules for multiplying integers. For a short introduction to visualizing repeated multiplication, you can place the transparency on the overhead and have a short discussion with students.

1. You could talk with students about why it is easier and quicker to remember the multiplication facts and to know how to use the calculator to multiply large numbers instead of repeated addition. Students may say that there is more room for mistakes in repeated addition with large numbers, it saves time to multiply instead of using repeated addition, and it takes up less room on my paper.

2. Students may say that it is good to use repeated addition when learning what multiplication means or to help explain it to someone who is just learning multiplication.

You may want to ask students, before they look at the lesson, if they think it will be possible to represent multiplication of negative numbers with repeated addition and manipulatives.

TEACHER’S NOTES

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________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
Using manipulatives is a method to understand multiplication. Study the following representations of multiplication then answer the two questions.

\[ 5 \cdot 3 = 15 \]
\[ 3 + 3 + 3 + 3 + 3 = 15 \]

1. What are the advantages of multiplication instead of repeated addition?

2. When is it better to use repeated addition?
**Discovery Activity**

The goal for this activity is for students to experience a way to understand the multiplication of integers. Students will use the red and yellow unit tiles to model integer multiplication. Make sure students actually build the repeated sets with tiles so that they construct a visual representation that they can recall in the future. If you have not already grouped your students, you can use the method below.

**Grouping Strategy**

Use a deck of cards and hand one card to each student. Modify the deck of cards for the number of students according to the rules below.

Have students get into groups of four using the following rules:

1. Each group must have one of each suit.
2. Each group must have only one of each number or value (one J, Q, K, or A).

Guide students through the activity's first page then direct them to complete Exercise 1. This will seem simple to students, but they need to experience the method so that it makes sense before moving forward. Have students share how they answered Exercise 1 at the overhead, then guide students through to Exercise 2. You may need to model a few examples of a negative number times a negative number before you have the groups of students work on Exercises 2 through 5. Use the parallel method described in Lesson 2 “Discovery Activity” to model examples. While you complete an example, students parallel you with their own problem. The most important exercise in this activity is Exercise 3. Make sure the groups of students spend time developing good rules. You may want to have students share their rules with the whole class and as a class develop a rule that works in general. In Exercise 4 students use calculators to confirm their rule. You can have them do this before or after you talk about Exercise 3 as a class.

*Optional Demonstration*

If you are proficient with a camera and video player, you could play a video of an object moving forward at a constant rate and then play it backward to watch it go back in time. You can discuss with students that the movie is about positive rate and positive time. Playing time backward is still about positive rate but since it is negative time the distance is negative, thus a positive times a negative is a negative.

End this activity with a short discussion. Tell students that forgetting procedures is easy to do. Generally, this can happen when we don't understand why the procedures work or when we don't have a way to link them to anything in our mind. One of the common things that many people forget in mathematics are the rules for multiplying integers. Place “Transparency 1” on the overhead to review with students the rules for multiplying integers and to look at a few examples. Ask the students if any of them have a way to remember these rules. Have the students who can remember the rules share their ideas with the class. If an idea that a student shares seems like a method that the rest of the class likes, you may want to make a poster of it and place it in the room.
Multiplication can be thought of as repeated addition. For example, $3 \cdot 5$ is the same as $5 + 5 + 5$ or $3 + 3 + 3 + 3 + 3$.

You could visualize $3 \cdot 5$ as 3 sets of 5 coins or 5 sets of 3 coins.

In either situation, there are 15 coins total!

Positive tiles could also be used to model multiplication of positive integers. For example,

```
3 Sets of 5
+ + +
+ + +
+ + +

5 Sets of 3
+ +
+ +
+ +
+ +
+ +
```

You can also think of multiplying a positive number and negative number as repeated addition and model it with two color tiles. For example, $3 \cdot (-5)$ is the same as $(-5) + (-5) + (-5)$ and can be modeled with 3 sets of 5 negative tiles.

```
3 Sets of -5
---
---
---
```

1. With your group, draw a visual model of the following expressions. Then, determine the product.

*Drawings will vary, but should include sets of numbers or tiles.*

- a. $2 \cdot 6 = 12$
  - 2 sets of 6 or 6 sets of 2

- b. $4 \cdot 3 = 12$
  - 4 sets of 3 or 3 sets of 4

- c. $2 \cdot (-7) = -14$
  - 2 sets of $-7$

- d. $(-4) \cdot 5 = -20$
  - 5 sets of $-4$

*Exercises “c.” and “d.” work easier when students start with the positive number as the number of sets.*
So how can we model a multiplication problem with two negative integers like this?

\[-3 \cdot (-5)\]

The idea is similar to \(3 \cdot (-5)\) but instead of placing 3 sets of \(-5\) tiles on the table, you need to take away 3 sets of \(-5\). We will need to use zero pairs to do this. Begin with a set of nothing, then represent the set of nothing with 3 sets of 5 zero pairs. Taking away the 3 sets of \(-5\) leaves 15 positive tiles for a solution of 15.

1. Set of nothing.

2. Three sets of five zero pairs.

3. Take away three sets of negative five.

4. There are 15 positive tiles left.

Therefore, \(-3 \cdot (-5) = 15\)

2. Use tiles to evaluate the following expressions. Write the product in the space provided. Include a drawing of the tiles. **Drawings will vary, but should include sets of numbers or tiles.**

a. \(-3 \cdot (-6) = 18\)

The drawing should represent taking away 3 sets of \(-6\). Students should start with 3 sets of six zero pairs. An alternative is to represent taking away 6 sets of \(-3\).

b. \(-7 \cdot (-2) = 14\)

The drawing should represent taking away 7 sets of \(-2\). Students should start with 7 sets of two zero pairs.

An alternative is to represent taking away 2 sets of \(-7\).
3. Based on what you discovered in this activity, create a multiplication rule for each situation below. *Sample responses are given. Good rules will describe both the sign and the product.*
   
   a. A positive number times a positive number.
      *To multiply a positive times a positive just multiply the numbers. The answer will be positive.*

   b. A positive number times a negative number.
      *To multiply a positive times a negative, multiply the number parts and the answer will be negative.*

   c. A negative number times a positive number.
      *To multiply a negative times a positive, multiply the number parts and make the answer negative.*

   d. A negative number times a negative number.
      *To multiply a negative times a negative, multiply the number parts and make the answer positive.*

We often use calculators to do the work for us, but other times we can use calculators to help us understand a concept, find a pattern, learn something new, or check a hypothesis.

4. First complete the following calculations without a calculator by using your integer multiplication rules. Then test your answers by completing the calculations with a calculator.

   **A Positive Times a Positive**
   
   a. \( 5 \times 8 = 40 \)  
   b. \( 7 \times 3 = 21 \)

   **A Positive Times a Negative**
   
   c. \( 6 \times (-5) = -30 \)  
   d. \( 8 \times (-3) = -24 \)

   **A Negative Times a Positive**
   
   e. \( -4 \times 7 = -28 \)  
   f. \( -9 \times 9 = -81 \)

   **A Negative Times a Negative**
   
   g. \( -6 \times (-3) = 18 \)  
   h. \( -8 \times (-4) = 32 \)

5. Explain how the calculator confirmed your rules. Did you need to adjust your rules? *Answers will vary. Students should find that the calculator confirmed their rules. Some groups of students may need to adjust their rules.*
Recalling the Rules for Multiplying Integers

\[(+)(+) = (+)\]
\[(+)(-) = (-)\]
\[(-)(+) = (-)\]
\[(-)(-) = (+)\]

Examples,

\[(3)(4) = 12\]
\[(3)(-4) = -12\]
\[(-3)(4) = -12\]
\[(-3)(-4) = 12\]
To understand the following story, think of a good guy as “positive (+)” and a bad guy as “negative (−).” Think of coming to town as positive (+) and leaving town as negative (−).

If a good guy (+) comes to town (+), then overall that is positive (+). \[(+)(+) = (+)\]

If a good guy (+) leaves town (−), then overall that is negative (−). \[(+)(-) = (-)\]

If a bad guy (−) comes to town (+), then overall that is negative (−). \[(-)(+) = (-)\]

If a bad guy (−) leaves town (−), then overall that is positive (+). \[(-)(-) = (+)\]

Adapted from 101+ Great Ideas for Introducing Key Concepts in Mathematics.
Math at Work

You may want to preface this “Math at Work” with a demonstration using a lever. Have students work in groups of three.

Assign each person a primary role in the group: A reader (who will read the material aloud), a writer (who will record work on the poster paper or transparency), and a presenter (who will present a problem to the class).

Symbolize It

In the previous lessons, students worked with the commutative and associative properties of addition. The students also determined that these properties did not work for subtraction. In this “Symbolize It” students will engage in the commutative and associative properties of multiplication as well as the distributive property. Have students work in pairs or groups to complete these exercises. In addition, a few problems have been added that include scalar multiplication of matrices.

TEACHER’S NOTES

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________________________________________________________________________________
________________________________________________________________________________
Have you ever played on a see-saw? Have you ever noticed how the see-saw changes depending on the weight of each person?

**Physics of a Lever**

Given a force, we determine the work by multiplying the force of the object by its distance to the **fulcrum**, which is the support or point of support on the see-saw. We can think of the fulcrum as the piece that is bolted to the ground.

To balance a see-saw, the total work (force \( \cdot \) distance) on each side must be equal. So for the see-saw above to be balanced, \( f_1 \cdot d_1 = f_2 \cdot d_2 \). For example, a 105 pound bear could sit 4 feet from the fulcrum on the left and a 140 pound bear could sit 3 feet from the fulcrum on the right to balance the see-saw.

**More Than Two Objects**

If there are more than two objects, then the sum of their work must be equal for the see-saw to balance. The equation to balance the see-saw below would be \( f_1 \cdot d_1 = f_2 \cdot d_2 + f_3 \cdot d_3 \).

1. Write the equation that represents this figure.

\[
20(5) = 30(2) + 10(4)
\]
Use the diagrams and background on see-saws to help you solve the following problems. Show all work or ideas leading to your solution.

2. Phoebe and Patrick are playing on a see-saw. Phoebe weighs 65 lbs and is sitting four feet from the fulcrum, and Patrick weighs 130 lbs. How far away should he sit from the fulcrum for the see-saw to be balanced?

   Patrick needs to sit 2 feet from the fulcrum. $65(4) = 130(2)$

3. Big Bear and Little Bear are playing on a see-saw. Little Bear is sitting 5 feet away and weighs 40 pounds, and Big Bear weighs 120 pounds and is sitting 3 feet away.

   You are going to lift Big Bear's end (apply a negative force), lifting 5 feet away from the fulcrum. How much force do you need to lift to keep the see-saw balanced?

   You'd need to lift by a force equal to $-32$ pounds. $40(5) = 120(3) + (-32)(5)$

4. Six weights have been placed on a see-saw in the following way:

   **Left Side**
   - 10 lbs 4 ft from fulcrum
   - 26 lbs 1 ft from fulcrum
   - 16 lbs 2 ft from fulcrum

   **Right Side**
   - 20 lbs 3 ft from fulcrum
   - 30 lbs 1 ft from fulcrum
   - 7 lbs 4 ft from fulcrum

   Sketch a diagram below. **Answers may vary, but sample shown below.**
Which way will the see-saw tilt? Explain your answer below.

The total work on the left is 98 and on the right is 118. The see-saw tilts down on the right side because there is more work on the right side pushing down.

You can apply a negative weight to the right side by placing a balloon that lifts in the opposite direction. Given the following three balloons, choose one negative weight to put on the see-saw to keep it balanced.

Which negative weight would you choose and how far from the fulcrum would you place the weight? Justify your answer using math, words, or both. Sample response: I would place the –10 pound balloon on the right side two feet from the fulcrum since –10(2) = –20 and 98 = 118 + (–20).
You may have learned some helpful ways to do mental math. We’re going to revisit some of the ways.

Suppose you had to multiply 5 and 13. You could either multiply by hand or you could try to multiply 13 by 5 in your head. How could you do that? $5 \cdot 13 = 5 \cdot 10 + 5 \cdot 3$  
$= 50 + 15$  
$= 65$

What did we just do? Instead of multiplying 5 by 13, we multiplied 5 by 10 and 5 by 3 and then added the products.

1. Try this example: $8 \cdot 12$
   
   $8 \cdot 12 = 8 \cdot 10 + 8 \cdot 2$  
   $= 80 + 16$  
   $= 96$

We could also do this with subtraction. Let’s look at $3 \cdot 29$.
   
   $3 \cdot 29 = 3 \cdot 30 - 3 \cdot 1$  
   $= 90 - 3$  
   $= 87$

2. Apply this method to evaluate the expressions below.

   a. $2 \cdot 39$  
      $2 \cdot 39 = 2 \cdot 40 - 2 \cdot 1$  
      $= 80 - 2$  
      $= 78$

   b. $9 \cdot 42$  
      $9 \cdot 42 = 9 \cdot 40 + 2 \cdot 9$  
      $= 360 + 18$  
      $= 378$

   c. $2 \cdot 27$  
      $2 \cdot 27 = 2 \cdot 20 + 2 \cdot 7$  
      $= 40 + 14$  
      $= 54$

We refer to this type of multiplying as the **Distributive Property** because multiplication distributes over addition or subtraction. A symbolic definition of this property is:

$$a(b + c) = ab + ac \quad \text{or} \quad a(b - c) = ab - ac$$
3. Describe the Distributive Property in your own words.

*Answers will vary.*

4. Give an example of the Distributive Property using numbers.

*Answers will vary.*

5. List other areas where the terms distributive or distribute are used outside of mathematics.

*Answers will vary. Students should list such things as “distribute food, distribute packages, distribution center, etc.”*

Suppose you have to multiply the three numbers 6, 4, and 5. You may decide to multiply 6 and 4 first to get 24 and then multiply by 5.

But there is an easier way! You could instead multiply 4 and 5 to get 20 and then multiply by 6. Multiplying 6 and 20 is much easier than multiplying 24 and 5.

In other words, you can rearrange the numbers to make friendly numbers.

6. Try the following examples. Multiply the following numbers and remember to rearrange them first to make friendly numbers.

   a. \( 8 \cdot 2 \cdot 5 \)
   
      \[
      8 \cdot 2 \cdot 5 = (2 \cdot 5) \cdot 8 \\
      = 10 \cdot 8 \\
      = 80
      \]

   b. \( 4 \cdot 8 \cdot 5 \)
   
      \[
      4 \cdot 8 \cdot 5 = (4 \cdot 5) \cdot 8 \\
      = 20 \cdot 8 \\
      = 160
      \]

   c. \( 45 \cdot 12 \)
   
      \[
      45 \cdot 12 = 9 \cdot (5 \cdot 6) \cdot 2 \\
      = 9 \cdot 30 \cdot 2 \\
      = 9 \cdot 60 \\
      = 540
      \]

   d. \( 15 \cdot 12 \)
   
      \[
      15 \cdot 12 = 5 \cdot 3 \cdot 3 \cdot 4 \\
      = (5 \cdot 4) \cdot (3 \cdot 3) \\
      = 20 \cdot 9 \\
      = 180
      \]

Of course, we aren’t always given such simple problems. We can also break a number into its factors to help us multiply. For example, in Exercise 6 c, when multiplying 45 and 12, we can rewrite 45 and 12 into factors of 9, 5, 6, and 2 and then multiply the friendly factors.
While you may not think about it, you have applied this concept when adding numbers. For example, when you add the numbers 37, 8, and 2 in your head, you probably do:

37 + 8 + 2 = 37 + 10 = 47

instead of:

37 + 8 + 2 = 45 + 2 = 47

7. Use this idea with the following problems.

a. 17 + 6 + 24  
   17 + 30  
   47

b. 16 + 23 + 4  
   23 + 16 + 4  
   23 + 20  
   43

In order to complete this type of multiplying or adding we are using a combination of the **Commutative Property** and **Associative Property**. The Associative Property is mathematically defined in addition as $a + (b + c) = (a + b) + c$ and in multiplication as $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. The Commutative Property is mathematically defined in addition as $a + b = b + a$ and in multiplication as $a \cdot b = b \cdot a$.

8. Describe the Associative and Commutative Properties in your own words.

   **Associative Property**

   *Answers will vary.*

   **Commutative Property**

   *Answers will vary.*

9. Give an example of each property using numbers.

   **Associative Property**

   *Answers will vary.*

   **Commutative Property**

   *Answers will vary.*
In Lesson 2, we saw that matrices could be used to record data in one place. The example was about a clothing store that recorded the number of items they had in stock. The following information is a list of in-stock items at the beginning of the winter coat sale in October.

**October Stock**

Small Jean Jackets: 12  
Medium Jean Jackets: 6  
Large Jean Jackets: 8

Small Winter Parkas: 18  
Medium Winter Parkas: 15  
Large Winter Parkas: 12

Small Fleece Coats: 6  
Medium Fleece Coats: 13  
Large Fleece Coats: 7

This list is organized into a matrix as shown below:

```
<table>
<thead>
<tr>
<th></th>
<th>Jean</th>
<th>Winter Parka</th>
<th>Fleece Coat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>12</td>
<td>18</td>
<td>6</td>
</tr>
<tr>
<td>Medium</td>
<td>6</td>
<td>15</td>
<td>13</td>
</tr>
<tr>
<td>Large</td>
<td>8</td>
<td>12</td>
<td>7</td>
</tr>
</tbody>
</table>
```

The company has decided to triple its inventory for a big Columbus Day sale. Using **scalar multiplication**, we can multiply the above matrix by 3.

\[
\begin{bmatrix}
12 & 18 & 6 \\
3 & 6 & 15 \\
8 & 12 & 7
\end{bmatrix}
\times
\begin{bmatrix}
3 & 12 & 3 & 18 & 3 & 6 \\
3 & 6 & 3 & 15 & 3 & 13 \\
3 & 8 & 3 & 12 & 3 & 7
\end{bmatrix}
= \begin{bmatrix}
36 & 54 & 18 \\
18 & 45 & 39 \\
24 & 36 & 21
\end{bmatrix}
\]

We see that in scalar multiplication, each entry in the matrix is multiplied by the scalar value. The resulting matrix, with column and row headings, would be:

```
<table>
<thead>
<tr>
<th></th>
<th>Jean</th>
<th>Winter Parka</th>
<th>Fleece Coat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>36</td>
<td>54</td>
<td>18</td>
</tr>
<tr>
<td>Medium</td>
<td>18</td>
<td>45</td>
<td>39</td>
</tr>
<tr>
<td>Large</td>
<td>24</td>
<td>36</td>
<td>21</td>
</tr>
</tbody>
</table>
```
Unit 2: Numbers and Integers

10. What does the resulting matrix represent? Fill in the blanks below.

Small Jean Jackets: 36
Medium Jean Jackets: 18
Large Jean Jackets: 24

Small Winter Parkas: 54
Medium Winter Parkas: 45
Large Winter Parkas: 36

Small Fleece Coats: 18
Medium Fleece Coats: 39
Large Fleece Coats: 21

11. Determine what would be in stock if the company multiplied its inventory by five instead of three for the big Columbus Day sale. Use scalar multiplication.

\[
\begin{bmatrix}
12 & 18 & 6 \\
5 & 6 & 15 & 13 \\
8 & 12 & 7
\end{bmatrix}
\cdot
\begin{bmatrix}
(5)12 & (5)18 & (5)6 \\
(5)6 & (5)15 & (5)13 \\
(5)8 & (5)12 & (5)7
\end{bmatrix}
= 
\begin{bmatrix}
60 & 90 & 30 \\
30 & 75 & 65 \\
40 & 60 & 35
\end{bmatrix}
\]

12. Complete the scalar multiplication for the following matrices.

a. \[
\begin{bmatrix}
4 & -5 & 2 \\
3 & 7 & 3 \\
9 & -2 & 8
\end{bmatrix}
\cdot
\begin{bmatrix}
12 & -15 & 6 \\
21 & 9 & 30 \\
27 & -6 & 24
\end{bmatrix}
= 
\begin{bmatrix}
108 & 12 & 54 \\
63 & 54 & 90 \\
27 & -6 & 24
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & -5 & -3 \\
-6 & -2 & 6 \\
0 & -7 & 9
\end{bmatrix}
\cdot
\begin{bmatrix}
-6 & 30 & 18 \\
12 & -36 & -42 \\
0 & 42 & -54
\end{bmatrix}
= 
\begin{bmatrix}
-6 & 30 & 18 \\
12 & -36 & -42 \\
0 & 42 & -54
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
10 & -1 & -1 \\
-10 & 10 & -1 \\
-1 & -1 & 10
\end{bmatrix}
\cdot
\begin{bmatrix}
-100 & 10 & 10 \\
10 & -100 & 10 \\
10 & 10 & -100
\end{bmatrix}
= 
\begin{bmatrix}
-1000 & 100 & 100 \\
100 & -1000 & 100 \\
100 & 100 & -1000
\end{bmatrix}
\]
Outcome Sentences

I really liked the ____________________________________________________________ method to remember how to multiply integers because ____________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________

Multiplying integers still confuses me when ____________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________

I would like to learn more about ______________________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________

If I could ask the teacher to clarify one thing about integers I would ask __________________
_____________________________________________________________________________________
_____________________________________________________________________________________

I am surprised that __________________________________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________

Lesson 5: Dividing Integers

In Brief
Students review the rules for dividing integers and use manipulatives to conceptually understand the integer rules for division.

Objectives
• Students represent integer division with tiles.
• Students understand the relationship between multiplication and division.
• Students calculate the division of integers with a calculator.
• Students understand integer division rules based upon real-world context.
• Students can remember and use integer rules.

Shaping the Lesson
• Various conceptual methods are given to help students understand and remember the rules for dividing integers.

Instructional Strategies
• Discussion
• Lecture
• Guided Practice
• Group Work
• Outcome Sentences

Tools
• Tiles
• “Setting the Stage” Transparency
• “Fact Family Template” Transparency
• Overhead Projector or other projection device

Warm Up
Problem of the Day
Setting of the Stage

Lead a discussion with the students asking them, "What method do you use to understand how division works?" Have a student volunteer record the students’ responses on the board. Place “Transparency 1” on the overhead. Have a student volunteer(s) read the transparency. Ask which students have used the displayed method. Have a student volunteer record the number of students who have used a grouping method to understand division. If the method wasn't in the list of students' responses, add it to the ones already on the board. For Exercise 1, have a student volunteer come to the board or overhead and draw a diagram. A sample answer could be 6 people with 2 pieces of pizza each. For Exercise 2, you may have to direct students to the commutative property of multiplication. For example, $12 = (2)(6)$ can be represented by 2 groups of 6 and $12 = (6)(2)$ can be represented by 6 groups of 2.

TEACHER’S NOTES
When you first learned to divide, you may have been presented with a **grouping** model.

For example, we can think of 12 divided by 2 as 12 items being organized into 2 equal groups. For example, if we divide 12 pieces of pizza between 2 people, each person gets 6 pieces.

\[
\frac{12}{2} = 6
\]

**Check:** We can use multiplication to check this division. Two people each have 6 pieces of pizza making a total of 12 pieces; \(12 = (2)(6)\).

1. We can also represent 12 divided by 2 as 12 items organized into groups of 2. How could you represent this with a drawing?

2. What multiplication property allows us to think this way?
Discovery Activity 1

Have students work in pairs. Students will need two sets of tiles for the second part of the activity.

Facilitate student cooperation by having students read through the models aloud within their pairs. Rotate among the pairs and encourage student thinking by asking guiding questions such as, "What is the difference between the two models? Which model do you use more often?" or "How is adding a negative number the same as subtracting a positive?"

After the class has completed Exercises 1 through 4 bring them back together and have student volunteers share their modeling results. Ask the students if they have a model preference or if both models work for them.
Modeling Dividing Integers Using Tiles

The grouping model of division with tiles will help us work with division of integers. Let's start with a division problem that we know. Let's start with 12 divided by 2.

We can think of this problem as, "How many groups of 2 will make 12?"

- Begin with zero unit tiles on your pad.
- Place one group of 2 positive unit tiles on your pad.
- Place another group of 2 positive unit tiles on your pad.
- Continue this process until you have 12 positive unit tiles on your pad.

Because it took 6 groups of 2, the solution to 12 divided by 2 is 6.

1. With a partner practice how you could represent 15 divided by 3 with unit tiles. Be prepared to present at the overhead. *Students should start with a blank pad and add groups of 3 until there are 15 tiles to show that 15 divided by 3 is 5.*
We can use this same method to divide $-10$ by $-2$. We can think of this problem as, "How many groups of $-2$ will make $-10$?"

2. Follow the steps given below to model the division with unit tiles. Draw diagrams that show your steps with the tiles and be prepared to present at the overhead.

Sample diagrams are shown below:

- Begin with zero unit tiles on your pad.

- Place one group of 2 negative unit tiles on your pad.

- Place another group of 2 negative unit tiles on your pad.

- Continue this process until you have 10 negative unit tiles on your pad.

How many groups of $-2$ make $-10$?

Because 5 groups of $-2$ make $-10$, then $-10$ divided by $-2$ is 5. $-10 \div (-2) = 5$

3. Use this method to model the following divisions

a. $-21 \div (-3)$

Because 7 groups of $-3$ make $-21$, then $-21$ divided by $-3$ is 7. $-21 \div (-3) = 7$

b. $-18 \div (-6)$

Because 3 groups of $-6$ make $-18$, then $-18$ divided by $-6$ is 3. $-18 \div (-6) = 3$
Unit 2: Numbers and Integers

We can use this method to divide 12 by –2. We can think of this problem as, "How many groups of –2 will make 12?" Because, negative numbers cannot be added to the tile pad to make a positive number, the problem actually becomes, "How many groups of –2 can you take away from a value of zero to make 12?"

1. **Begin with zero tiles on your pad.**

   \[
   12 \div (-2) \]

2. **Place 12 zero pairs on the tile pad. Keep the zero pairs in groups of two. The pad still represents a value of zero.**

3. **Take one group of 2 negative unit tiles off your pad.**

4. **Continue this process until you have 12 positive algebra tiles on your pad.**

   \[
   12 \div (-2) = -6
   \]

5. **Because you had to take 6 groups of –2 away, the solution to 12 divided by –2 is –6. It takes –6 groups of –2 to make 12.**

4. **With a partner practice how you could represent 15 divided by –5 with tiles. Be prepared to present at the overhead.** 

   **Students should start with a blank pad and add 15 zero pairs. Then take away groups of –5 until there are 15 positive tiles left. Because you have to take away 3 groups of –5 the solution is –3.**
Let’s look at 12 divided by 2 with unit tiles in another way. We can think of this problem as, "12 can be divided into 2 equal groups of what size?"

- Start with 12 positive unit tiles on the tile pad.

  ![Image of 12 positive unit tiles]

- Arrange the 12 tiles into 2 groups of equal size.

  ![Image of 12 tiles divided into 2 groups]

- Because there are 6 tiles in both groups, the solution to 12 divided by 2 is 6.

We can use this same method to divide –12 by 3. We can think of this problem as, "–12 can be divided into 3 equal groups of what size?"

5. Follow the steps given below to model the division with unit tiles. Draw diagrams that show your steps with the tiles and be prepared to present at the overhead.

- Begin with 12 negative unit tiles on your pad.

  ![Image of 12 negative unit tiles]

- Arrange the 12 negative unit tiles into three groups of equal size.

  ![Image of 12 negative unit tiles divided into 3 groups]

  a. –12 can be divided into 3 equal groups of what? Because –12 can be divided into 3 equal groups of –4, then –12 divided by 3 is equal to –4. \( -12 \div 3 = -4 \).

6. Use the different methods to model the following divisions. **Modeling answers will vary**

   a. \(-6 \div 2 = -3\)  
   b. \( -9 \div (-3) = 3 \)  
   c. \( 14 \div (-2) = -7 \)

7. Write a rule below for the four different combinations of integers. **Answers will vary. Sample response:** Ignore the sign and just divide the numbers then follow these rules: A positive divided by a positive is a positive, a positive divided by a negative is a negative, a negative divided by a positive is a negative, and a negative divided by a negative is a positive.
Discovery Activity 2

Show “Transparency 1.” Make sure to cover the lower half that deals with the two exercises and dividend, divisor, and quotient relationships. Have a student volunteer read the first paragraph. Have a student volunteer explain how the division equation and multiplication problem are related. A sample response may be, “You know that 15 divided by 3 is 5 because 3 times 5 is 15.” Reveal Exercises 1 and 2. Give the students 30 seconds to think of a response then use “Pass It Along” to have two students give a response. Finally, reveal the dividend, divisor, and quotient relationships and complete a short discussion on these relationships. Discuss with students that the divisor and the quotient are just the factors of a multiplication problem and that the dividend is the product of the related multiplication problem. An example is given below that may help students see the relationship between the dividend (product) and the divisor and the quotient (factors).

\[
\frac{\text{dividend}}{\text{divisor}} = \text{quotient} \quad \quad \quad \frac{\text{product}}{\text{factor}} = \text{factor}
\]

\[
\text{dividend} = (\text{divisor})(\text{quotient}) \quad \quad \text{product} = (\text{factor})(\text{factor})
\]

Show “Transparency 2.” Cover the bottom card. Let students know they will create a family fact to relate the parts of a division problem to the parts of a multiplication problem. Show students how the card works. If you start at the top, go counterclockwise to the lower left, and continue counterclockwise to the lower right, the matching division problem would be $-35$ divided by $7$ is $-5$. Or, if you start at the lower left, go counterclockwise to the right, and continue counterclockwise to the top, the matching multiplication problem is $7$ times $-5$ is $-35$. The versatility of using this type of card is that a person can go either clockwise or counterclockwise to create many different multiplication and division family facts. You may want to encourage students to determine each family fact that could be created with the card. For example,

- **Counterclockwise**: $7(-5) = -35 \quad -35 ÷ 7 = -5$
- **Clockwise**: $-5(7) = -35 \quad -35 ÷ (-5) = 7$

Have each student make his or her own family fact card with integers. Cutouts are available at the end of the lesson in the Student Journal. Make sure that students include negative numbers. For practice you can have students share their family fact cards with others as you might use flash cards.

You can use Transparencies 3 and 4 to review the multiplication rules with the class. Extend the discussion about the relationship between multiplication and division to the sign rules for division and multiplication.

Reinforce with the students that we can use multiplication to check a division problem and division to check a multiplication problem.
Another method to understand division is with the corresponding multiplication. We can use our multiplication facts to determine the answer to a division problem. For example,

\[
\frac{15}{3} = 5 \\
15 = (3)(5)
\]

Use this method to determine the following divisions.

1. \[\frac{-21}{7} = \quad \]
2. \[\frac{18}{-3} = \quad \]

The relationship between division and multiplication can be seen with the following descriptions:

\[
\frac{\text{dividend}}{\text{divisor}} = \text{quotient}
\]

\[
\text{dividend} = (\text{divisor})(\text{quotient})
\]
Let’s use this multiplication and division relationship to create family fact cards.

\[ \text{Dividend} \]

\[ \div = \quad \div = \]

\[ 7 \quad \bullet = \quad -5 \]

\[ \text{Divisor or Quotient} \quad \text{Divisor or Quotient} \]
Recalling the Rules for Dividing Integers

\[
\frac{(+)}{(+)} = (+) \quad \frac{(+)}{(-)} = (-) \\
\frac{(-)}{(+)} = (-) \quad \frac{(-)}{(-)} = (+)
\]

Example,

\[
\frac{12}{4} = 3 \text{ because } 12 = (4)(3)
\]

\[
\frac{-12}{3} = -4 \text{ because } -12 = (3)(-4)
\]

\[
\frac{-12}{4} = -3 \text{ because } -12 = (4)(-3)
\]

\[
\frac{12}{-4} = -3 \text{ because } 12 = (-4)(-3)
\]
Remember the Rules by Relating Them to a Story.

Remember the story from the multiplication lesson. In division, the good or bad guy is already in town. A good guy is positive (+) and a bad guy is negative (−). Staying in town is positive (+) and leaving town is negative (−).

If a good guy (+) stays in town (+), then overall that is positive (+).

\[
\frac{(+)}{(+) = (+)}
\]

If a good guy (+) leaves town (−), then overall that is negative (−).

\[
\frac{(+)}{(-) = (-)}
\]

If a bad guy (−) stays in town (+), then overall that is negative (−).

\[
\frac{(-)}{(+) = (-)}
\]

If a bad guy (−) leaves town (−), then overall that is positive (+).

\[
\frac{(-)}{(-) = (+)}
\]

Adapted from 101+ Great Ideas for Introducing Key Concepts in Mathematics.
# Lesson 5

## Unit 2: Numbers and Integers

### Symbolize It

**SJ Page 43**

1. Evaluate each expression below, using tiles if needed.
   
   a. \(-9 ÷ 3 = -3\)  
   b. \(-9 ÷ (-3) = 3\)  
   c. \(-10 ÷ (-5) = 2\)  
   d. \(-18 ÷ 6 = -3\)  
   e. \(-12 ÷ (-4) = 3\)  
   f. \(-16 ÷ (-4) = 4\)  
   g. \(-20 ÷ (-5) = 4\)  
   h. \(-21 ÷ (-3) = 7\)  
   i. \(-18 ÷ (-9) = 2\)  
   j. \(35 ÷ (-5) = -7\)

2. Use prior knowledge to solve \(-28 ÷ (-4)\).  
   \[ 7 \]

3. For the problems below, fill in the blank.
   
   a. \(36 ÷ \underline{9} = -4\)  
   b. \(-5 \underline{(-3)} = 15\)  
   c. \(\underline{28} ÷ (-7) = -4\)

4. Use your calculator for the following division problems.
   
   a. \(-135 ÷ (-9) = 15\)  
   b. \(225 ÷ (-15) = -15\)  
   c. \(-231 ÷ 3 = -77\)

5. Match the division terms on the left with their corresponding multiplication term on the right.

<table>
<thead>
<tr>
<th>Division Terms</th>
<th>Multiplication Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividend</td>
<td>Factor</td>
</tr>
<tr>
<td>Divisor</td>
<td>Factor</td>
</tr>
<tr>
<td>Quotient</td>
<td>Product</td>
</tr>
</tbody>
</table>
Tile Pad
Outcome Sentences

I really liked the ____________________________ method to remember how to divide integers because ____________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

Dividing integers still confuses me because ____________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

I would like to learn more about ____________________________
_______________________________________________________________________________________
_______________________________________________________________________________________

If I could ask the teacher to clarify one thing about integers I would ask ________________
_______________________________________________________________________________________
_______________________________________________________________________________________
Lesson 6: Natural Numbers to Rational Numbers

**In Brief**
Students conclude that we need rational numbers because integers divided by integers do not always produce integers. They also review the sign rules of integers to determine if the rules also apply to rational numbers.

**Objectives**
- Students will determine that the set of integers is not closed for division.
- Students will determine that the set of rational numbers is closed under division for all rational numbers with the exception of dividing by zero.

**Shaping the Lesson**
- Students investigate different number systems to determine which are closed under addition, subtraction, multiplication, and division.

**Instructional Strategies**
- Pass it Along
- Read, Reflect, Rephrase

**Tools**
- “Setting the Stage” Transparency
- “Grouping” Transparency
- Overhead Projector or other projection device

**Warm Up**
Problem of the Day
Setting of the Stage

Show the “Setting the Stage” transparency. This activity’s purpose is for students to experience various situations where rational numbers are intuitive. Because the purpose of this lesson is to engage students with rational numbers, let students use their own vocabulary and intuition. The questions are open-ended, so be sure to ask guiding questions to assess students prior knowledge. This part of the lesson is intended to be brief (10 minutes). Have students work with a partner first on the four scenarios. Have one partner read the scenario aloud and then determine what you would do in this situation if you could use only integers. Then determine what you would do if you could use rational numbers. After approximately 2 minutes have pairs share their responses by using Pass it Along.

Pass it Along: Give a student a turn to respond to a question, or if a student prefers not to respond he/she can say, “I pass.” You can clarify the number of times that a student can pass. For example, you may want a student to pass no more than three times with the fourth person having to respond. You can have the passing follow the pattern of seating arrangements or you may want the student to determine whom he/she passes to.

Sample responses are given below.

Situation #1
- **Integers:** Three of them could pay $10.00 and two of them could pay $9.00.
- **Rational Numbers:** They could each pay $9.60.

Situation #2
- **Integers:** Five of the children get a candy bar and the sixth gets no candy bar or no one gets a candy bar.
- **Rational Numbers:** Each child gets 5/6 of a candy bar.

Situation #3
- **Integers:** Two of the friends get 3 pieces and one friend gets 2 pieces.
- **Rational Numbers:** Each of the friends gets two pieces and they split the last two pieces, so they each get 2 and 2/3 pieces.

Situation #4
- **Integers:** Each person fills 4 boxes and one person fills the extra box. This means three people fill 4 boxes and one person fills 5 boxes.
- **Rational Numbers:** Each person fills 4 and 1/4 boxes.
Integer Solution?

Rational Number Solution?

Situation #1
Five roommates decide to get internet cable. The monthly charge is $48.00. How much money should each person pay each month?

Situation #2
There are five candy bars left. There are six children who want a candy bar. How much candy will each child get?

Situation #3
Three friends order a pizza. There are 8 pieces of pizza. How many pieces will each friend get?

Situation #4
You and your friends have volunteered to box food for a local shelter. There are 17 boxes to fill and there are 4 of you. How many boxes will each fill?
Guided Discussion/Partner Reading

Use the worksheet as a guide for a discussion with students about rational numbers. Note that the next unit on rational numbers assumes that students are familiar with the term, rational numbers. The purpose of this discussion is to understand that each set of numbers discussed is contained in the subsequent set. In other words, the natural numbers are a subset of the whole numbers which are a subset of the integer numbers which are a subset of the rational numbers.

Following the guided discussion, have students read through the narrative entitled History of Rational Numbers. You may want to use the reading strategy, Read, Reflect, Rephrase (described below). This part of the lesson should last 15 minutes.

Read, Reflect, Rephrase – Read (or have a student read) a short selection (several sentences) or paragraph. Have the students think about what was said and then rephrase or restate what was read. This can be done as a class or in small groups.

TEACHER’S NOTES

________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
What exactly are rational numbers?
When we think of rational numbers, we may think about decimals and fractions. While these are both ways to represent rational numbers, it is important that we understand what we really mean by the term, rational numbers.

Formal Definition
The set of rational numbers is the set of all possible ratios of integers without zero in the denominator.

A rational number can be expressed as the fraction \( \frac{a}{b} \) where \( a \) and \( b \) are integers and \( b \neq 0 \).

What does this mean?
If you divide an integer by another integer, the quotient is a rational number. We can express this quotient as a fraction or decimal. The set of possible quotients is called the rational number set.

1. Are natural numbers included in the rational number set? Explain below why or why not.
   Yes, because we can write any natural number as itself divided by 1. For example, 3 can be written as 3/1.

2. Are whole numbers included in the rational number set? Explain below why or why not.
   Yes, because we can also write 0 as 0/1.

3. Are integers included in the rational number set? Explain below why or why not.
   Yes, because any integer can be written as itself divided by 1. For example, –4 can be written as –4/1.

4. Why do we sometimes associate rational numbers with fractions?
   Answers will vary. The purpose of this question is to prompt students to think about why they have associated the term fraction with rational numbers; this is to help solidify student understanding about rational numbers.
History of Rational Numbers

The ancient Egyptians may have been the first groups of people to write fractions. They used a special symbol to represent a numerator of one. Examples are shown below.

\[
\begin{align*}
\text{\[\begin{array}{c}
\text{\(\frac{1}{3}\)}
\end{array}\]}
\text{\[\begin{array}{c}
\text{\(\frac{1}{10}\)}
\end{array}\]}
\end{align*}
\]

Because only fractions with a numerator of 1 could be written, writing other fractions and performing calculations involving fractions were tedious and time-consuming.

The Babylonians (who lived in what is now Iraq) had a number system similar to ours except it was based on 60 instead of 10. They devised a method of writing both the numerator and denominator to represent fractions. For example, \(3\frac{5}{60}\) would be written as a 3 followed by a 5. This is particularly confusing since the Babylonians would also write a 3 followed by a 5 to represent \(3 \times 60^2 + 5 \times 60\). The Babylonians would have to use the context of the problem to know which number was represented.

\[
\begin{align*}
\text{\[\begin{array}{c}
\text{\(3\frac{5}{60}\)}
\end{array}\]}
\end{align*}
\]

Not having a decimal point, it was difficult to determine what place each number represented. For example, imagine if you saw the number 35 and you were told the decimal point was missing. You might have the number 35.0, 3.5, or 0.35.

The notation of ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and of writing the numerator over the denominator developed in India more than 1500 years ago. Historical records show that the fraction bar was used regularly by Leonardo of Pisa, also known as Fibonacci, during the 13th century and that the bar was also used before that in Arabia. It wasn't until the 16th century that the fraction bar was used in general.
**Discovery Activity**

The idea of closure was mentioned in Lesson 1. The purpose of this activity is for students to work with the idea of closure in understanding the place of rational numbers. Students will also investigate why a rational number cannot have zero in the denominator.

Have students work in groups of four or five so that there are six groups. A grouping strategy is given below.

Have students work through the discovery activity. Be sure to rotate among the groups and keep students on track. You want the students to work on metacognition (thinking about their thinking). Have students present their answers on poster paper. Have each group present the answer to one portion (so that two groups will present an answer to each part of the discovery activity). You may want faster working groups to present parts 2 or 3 and slower groups to present parts 1 or 2.

**Grouping Strategy**

This strategy was prepared for a class of 36 (i.e. six groups of six), so you will need to modify for your classroom. Cut out the numbers on the “Grouping Strategy” cut out page located at the end of the lesson. Give one number to each student. *(Note: If you do not want to cut the page out of your manual, photocopy the page and cut out the numbers.)* Place the transparency on the overhead and tell students they need to get into groups that satisfy the criteria listed. As each group is formed, check that it meets the criteria. You will need to firmly reassign students as they may be reluctant to work with others. If this seems too difficult for your students, make an overhead of the cutouts (before you cut them out!) and have students get into groups by row.

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**TEACHER’S NOTES**
Discovery Activity: Revisiting Closure

In Lesson 1, you learned about a concept called closure under an operation. For example, if a set is closed under addition, the sum of any two numbers in the set is also in the set.

**Whole numbers are closed under addition and multiplication.**
The set of whole numbers is \(\{0, 1, 2, 3, \ldots\}\). If you add any two numbers in this set, their sum is also in this set. The product of any two whole numbers is also a whole number.

**You try it:** Choose any two whole numbers. Add them, and the sum is also a whole number. Multiply them, and the product is also a whole number.

**Whole Numbers are NOT closed under subtraction and division.**
If you subtract two whole numbers, you may not get a whole number. The same is true for division. If you divide two whole numbers, you may not get a whole number.

**You try it:** Choose any two whole numbers. Subtract the smaller number from the larger number. Subtract the larger number from the smaller number. Did you get a whole number both times?

**You try it:** Choose any two whole numbers. Divide the larger number by the smaller number. Divide the smaller number by the larger number. Did you get a whole number both times?

1. **When is the set of integer numbers closed?**
   Determine if the set of integers is closed under the operations of addition, subtraction, multiplication, division. Record your answer in the space below. Be prepared to present your answer to the class.
   
   **Hint:** Use guess and check methods similar to what you did for the whole numbers.

   The set of integers will be closed under addition, subtraction, and multiplication.
   The set of integers is not closed under division.

2. **When is the set of rational numbers closed?**
   Determine if the set of rational numbers is closed under addition, subtraction, multiplication, and division. Record your answer in the space below. Be prepared to present your answer to the class.
   
   **Hint:** Use guess and check methods similar to what you did for the whole numbers.

   The set of rational numbers is closed under addition, subtraction, and multiplication. The set of rational numbers is not closed under division only because of dividing by zero.
Why is it impossible to divide by zero?
We’ve seen that none of the sets of numbers are closed under division, namely because a number cannot be divided by zero. The purpose of this part of the discovery activity is to understand why the rational numbers are not closed under division.

Relationship between Multiplication and Division
To help us understand why we cannot divide by zero, we need to understand the relationship between multiplication and division. Recall the family facts that you worked with in lesson 5.

\[
\frac{\text{dividend}}{\text{divisor}} = \text{quotient} \quad \frac{\text{dividend}}{\text{factor}} = \text{factor}
\]

\[
\text{dividend} = (\text{divisor})(\text{quotient}) \quad \text{dividend} = (\text{factor})(\text{factor})
\]

1. Finish the multiplication and division family facts for the equations below.
   \[12 = 3 \cdot 4\] \[12 \div 3 = 4\]

2. Write a sentence describing the relationship between multiplication and division.

   *Answers will vary.*

Now that we understand how multiplication and division are related, we will see why we can’t divide by zero.

3. Use the multiplication and division family facts to fill in the equation below.
   \[5 = \text{ } \cdot 0\] \[5 \div 0 = \text{ }\]

   *Answer will vary.*

What do you notice about these equations? Record any other thoughts below. Be prepared to present this argument to the class.

*Answers will vary. The goal is for students to see that to divide five by zero would be equivalent to multiplying zero by another number to result in five. Because zero times anything is zero, the students should be able to state that it is impossible to complete these equations.*
### Grouping Cutouts

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2/3</td>
<td>-5/7</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>-6</td>
</tr>
<tr>
<td>-2/3</td>
<td>-2.5</td>
<td>5</td>
<td>0</td>
<td>-1</td>
<td>-11</td>
</tr>
<tr>
<td>8.7</td>
<td>-3/4</td>
<td>8</td>
<td>0</td>
<td>-5</td>
<td>-12</td>
</tr>
<tr>
<td>-1.2</td>
<td>1/4</td>
<td>3</td>
<td>0</td>
<td>-7</td>
<td>-9</td>
</tr>
<tr>
<td>5/6</td>
<td>-1/10</td>
<td>2</td>
<td>0</td>
<td>-4</td>
<td>-10</td>
</tr>
<tr>
<td>-7/8</td>
<td>4.5</td>
<td>4</td>
<td>0</td>
<td>-8</td>
<td>-13</td>
</tr>
</tbody>
</table>
Find other students to form a group that satisfies the following criteria:

- Has exactly 1 natural number
- Has exactly 2 whole numbers
- Has exactly 4 integers
- Has at least 5 rational numbers

Once you’ve found your group, raise your hand so that your teacher can check. If your group does not satisfy the criteria, you must find new group members.
**Math at Work**

Investigating Rules for Signed Rational Numbers *

This is meant to be a short activity that bridges the lessons from the unit. Rotate among groups to help students create rules for signed rational numbers. Have students use calculators and brainstorm with various rational numbers.

Students should describe the rules for adding, subtracting, multiplying, and dividing that they learned in the unit.

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**TEACHER’S NOTES**

__________________________________________________________

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__________________________________________________________

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__________________________________________________________

__________________________________________________________
Investigating Rules for Signed Rational Numbers

Throughout this unit, you have discovered the sign rules for adding, subtracting, multiplying, and dividing integers. Record these sign rules below.

- Rule for addition

- Rule for subtraction

- Rule for multiplication

- Rule for division

Work in your group to answer the following question:

**Are the rules listed above the same for rational numbers?**

Hint: Use simple rational numbers to test your rules and your calculator to help with computations.

Complete each blank with either the words positive or negative.

1. The sum of $-4.5$ and $-1.3$ is ____ **negative** ____.
2. The product of $\frac{3}{2}$ and $-4$ is a ____ **negative** ____ number.
3. The quotient of $-10.5$ and $-3.2$ is a ____ **positive** ____ number.
4. The answer to $\frac{1}{4} - \left( \frac{1}{2} \right)$ is a ____ **negative** ____ number.
5. The sum of $\frac{-10}{3}$ and $\frac{3}{2}$ is ____ **negative** ____.
6. The product of $3.7$ and $-2.6$ is ____ **negative** ____.
7. The answer to $-14 - 6.6$ is a ____ **negative** ____ number.
Presenting Number Sets

Directions: You have learned about several sets of numbers. Work in your groups to prepare a five-minute presentation. Use poster paper or transparencies for your presentation. Your group presentation must answer the following questions:

• What are the different types of sets of numbers?
• How are these sets of numbers related?

Your presentation will be judged on the following criteria:

• Did all group members speak during the presentation?
• Did the group address all of the sets of numbers?
• Did the group compare the sets of numbers?
• Did the group include a visual aid?
• Extra points for originality and creativity.
1. Using a graphic organizer, draw the relationship between natural numbers, whole numbers, integers, and rational numbers.

2. Are the set of integers closed under multiplication and division? Explain.

3. Give an example of the Associative Property of Multiplication using integers.

4. Give an example of the Commutative Property of Addition using integers.

5. Give an example of the Distributive Property using integers.
6. Simplify each expression below.

a. \(12 + (-6)\)  
b. \(-23 + (-12)\)  
c. \(8 + (-8)\)  
d. \(-14 + 31\)  
e. \(-12 - (6)\)  
f. \(9 - (-7)\)  
g. \(-15 - (-4)\)  
h. \(22 - (-19)\)  
i. \(-15(-4)\)  
j. \(-7(14)\)  
k. \(-12(8)\)  
l. \(20(13)\)  
m. \(-45\)  
\[
\frac{-5}{-11}
\]  
o. \(\frac{121}{-11}\)  
p. \(\frac{99}{11}\)  

7. Given a force, we determine the **work** by multiplying the force of the object by its distance to the **fulcrum**, which is the support or point of support on the see-saw. We can think of this as the piece that is bolted to the ground.

![Diagram of a see-saw showing force and distance](image)

To balance a see-saw, the total work \((force \cdot distance)\) on each side must be equal. The equation to balance the see-saw above would be \(f_1 \cdot d_1 = f_2 \cdot d_2 + f_3 \cdot d_3\).

a. Six weights and a negative upward force have been placed on a see-saw in the following way:

![Diagram of a see-saw with weights and a negative upward force](image)

The table below represents each weight’s distance, from the fulcrum, and how much it weighs, along with the distance from the fulcrum where a negative upward force would need to be applied to balance the see-saw.

<table>
<thead>
<tr>
<th><strong>Left Side</strong></th>
<th><strong>Right Side</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>A: 12 lbs 4 ft from fulcrum</td>
<td>D: 28 lbs 1 ft from fulcrum</td>
</tr>
<tr>
<td>B: 10 lbs 2 ft from fulcrum</td>
<td>E: 15 lbs 3 ft from fulcrum</td>
</tr>
<tr>
<td>C: 45 lbs 1 ft from fulcrum</td>
<td>F: 7 lbs 4 ft from fulcrum</td>
</tr>
<tr>
<td>Upward Force: 3 ft from fulcrum</td>
<td></td>
</tr>
</tbody>
</table>

How much of a negative upward force must be applied to balance the see-saw?
b. Seven weights, including a negative weight balloon, have been placed on a see-saw in the following way:

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: 5 lbs 6 ft from fulcrum</td>
<td>D: 27 lbs 1 ft from fulcrum</td>
</tr>
<tr>
<td>B: 8 lbs 3 ft from fulcrum</td>
<td>E: 21 lbs 3 ft from fulcrum</td>
</tr>
<tr>
<td>C: 30 lbs 2 ft from fulcrum</td>
<td>F: 12 lbs 5 ft from fulcrum</td>
</tr>
<tr>
<td>Balloon: –18 lbs ? ft from fulcrum</td>
<td></td>
</tr>
</tbody>
</table>

How far from the fulcrum must the balloon be to balance the see-saw?

8. You are a teacher's aide in a mathematics classroom. One of the students is having a hard time understanding addition and subtraction of signed integers. Use tiles or number lines to model, to the student, the concepts of addition and subtraction with signed integers.

Pick two of the following expressions below and draw a sketch of how you could use tiles to evaluate the expression.

a. –6 + 9  
b. –4 + (–7)  
c. 8 – (–5)  
d. –7 – 6  
e. –8 – (–5)

Use the space below to draw the tiles.

Or, use the number lines below.
9. You are a meteorologist at the local TV station. Each day you read the satellite weather maps provided to you by NOAA (National Oceanic and Atmospheric Administration). For each scenario below, write the requested expression and then simplify to find the results.

a. The current temperature is displayed by the thermometer in Figure 1. From reading the satellite weather maps, you have determined that the temperature will drop 28°F during the night. Write an addition or subtraction expression and then simplify to determine the low temperature for the night that you will report during the 11 p.m. news.

b. The current temperature is displayed by the thermometer in Figure 2. From reading the satellite weather maps, you have determined that the temperature will rise 35°F. Write an addition expression and then simplify to determine the high temperature for the day tomorrow that you will report during the 11 p.m. news.

c. The owner of the TV station wants you to take periodic temperature readings. The temperature is going to drop 30°F during the night at a constant rate. The owner wants you to take readings every time the temperature drops 3°F. Write a division equation to represent the number of periodic readings you will have to take. Your equation should take into account that a drop in temperature would be recorded as a negative number.

d. While reading the satellite maps provided to you by NOAA, you have determined that the barometric pressure (atmospheric pressure as indicated by a barometer) is going to drop by an amount unprecedented before in the history of recorded weather in your area. You took temperature readings each time the temperature dropped 2°F and you took 34 readings. Write a multiplication equation to represent the amount of temperature drop and find the drop in temperature that was recorded. Your equation should take into account that a drop in temperature would be recorded as a negative number.
Unit 2 Individual Assessment – Teacher’s Key

1. Answers will vary.

2. The set of integers is closed under multiplication. The set of integers is not closed under division.

3. Sample response: $-4(2 \cdot 3) = (-4 \cdot 2)^3$

4. Sample response: $2 + (-5) = -5 + 2$

5. Sample response: $-6(-3 + 2) = -6(-3) + -6(2)$

6. a. 6  b. -35  c. 0  d. 17  e. -18  f. 16  
g. -11  h. 41  i. 60  j. -98  k. -96  l. 260  
m. 9  n. -8  o. -11  p. 9

7. a. -4 pounds: $12(4) + 10(2) + 45(1) + (?)(3) = 28(1) + 15(3) + 7(4)$ 
   \[113 + (?)(3) = 101\]
   \[? = -4\]
   b. 2 feet: $5(6) + 8(3) + 30(2) = 27(1) + 21(3) + 12(5) + (-18)(?)$
   \[114 = 150 + (-18)(?)\]
   \[? = 2\]

8. Answers will vary. Check for student understanding of using manipulatives to represent adding or subtracting integers.

9. a. $10 - 28 = -14$
   b. $-15 + 35 = 20$
   c. 10 periodic readings $-30 \div (-3) = 10$
   d. $-68^\circ F: -2(34) = -68$
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